19. *Finitely Additive Measures on* N

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1. Introduction. In this paper, we improve the theorem of Jech and Prikry [2] on projections of finitely additive measures. Let N denote the set of all natural numbers. A (finitely additive) measure on N is a function $\mu: P(N)\to[0,1]$ such that $\mu(\phi)=0$, $\mu(N)=1$ and if X and Y are disjoint subsets of N, then $\mu(X\cup Y)=\mu(X)+\mu(Y)$. μ is non-principal if $\mu(E)=0$ for every finite set $E\subset N$. Let $F: N\to N$ be a function. If μ is a measure on N, then $\nu=F^*(\mu)$ (the projection of μ by F) is the measure defined by $\nu(X)=\mu(F^{-1}(X))$.

Theorem (Jech and Prikry). *There exist a measure μ on N and a function $F: N\to N$ such that*

a) $F^*(\mu)=\mu$,

b) *if $X\subseteq N$ is such that F is one-to-one on X, then* $\mu(X)\leqq\dfrac{1}{2}$.

A measure is two-valued if the values is $\{0,1\}$. The theorem of Jech and Prikry contrasts with the following theorem concerning two-valued measure (Frolik [1] and Rudin [3]):

If μ is a two-valued measure and $F: N\to N$ is such that $F^*(\mu)=\mu$, then $F(x)=x$ on a set of measure 1.

In this paper we prove the following

Theorem. *There exist a measure μ and a function $F: N\to N$ such that*

a) $F^*(\mu)=\mu$,

b) *if $X\subseteq N$ is such that F is one-to-one on X, then* $\mu(X)=0$.

2. Sketch of the proof. We shall now state two results, to be proved in the following sections. We shall indicate how Theorem follows from them.

Proposition 1. *For any prime p, there exist a function $F_p: N\to N$ and a finitely additive measure η_p such that*

1) $F_p^*(\eta_p)=\eta_p$,

2) *if $X\subseteqq N$ is such that F_p is one-to-one on X, then* $\eta_p(X)\leqq 1/(p-1)$.

Proposition 2. *There exists a function $f_p: N\xrightarrow[\text{onto}]{1;1}N$ such that $f_pF_3^{-1}=F_p^{-1}f_p$ where F_3 and F_p are the functions in Proposition* 1.

We let $F=F_3$ and $\lambda_p(X)=\eta_p(f_p(X))$ where $f_p(X)=\{f_p(x)|x\in X\}$.

Since f_p is one-to-one and onto, λ_p is a finitely additive measure.

First we prove

3) $F^*(\lambda_p)=\lambda_p$,

4) if $X\subseteqq N$ is such that F is one-to-one on X, then $\lambda_p(X)\leqq 1/(p-1)$.

Since f_p is one-to-one and onto, 4) holds by 2) because if F is one-to-one on X, then F_p is one-to-one on $f_p(X)$. By 1), for any $X\subseteqq N$, $\eta_p(X)=\eta_p(F_p^{-1}(X))$. Therefore $\lambda_p(X)=\eta_p(f_p(X))=\eta_p(F_p^{-1}(f_p(X)))$ $=\eta_p(f_p(F^{-1}(X)))=\lambda_p(F^{-1}(X))$ by Proposition 2. Then 3) follows. It is important that in 3) and 4) F does not depend on p.

Let $\{a_n|n\in N\}$ be a bounded sequence of real numbers, and ν be a two-valued measure. Then there exists a unique real number a, which we denote by $a=\lim_\nu a_n$, such that for any $\varepsilon>0$, $\nu(\{n||a-a_n|<\varepsilon\})=1$.

Let p_n be the n-th prime number. By letting $\mu(X)=\lim_\nu \lambda_{p_n}(X)$, we get a theorem. Because μ is obviously a finitely additive measure, $\mu(X)=\lim_\nu \lambda_{p_n}(X)=\lim_\nu \lambda_{p_n}(F^{-1}(X))=\mu(F^{-1}(X))$ and if F is one-to-one on X, then $\mu(X)=\lim_\nu \lambda_{p_n}(X)\leqq \lim_\nu 1/(p_n-1)=0$.

3. Proof of Proposition 1. Original idea is due to Jech and Prikry. For each $X\subseteqq N$, we define $X(n)=$"the number of elements of $X\cap\{1,2,3,\cdots,n\}$" and $\mu_0(X)=\lim_\nu X(n)/n$. Obviously $\mu_0(X)=\mu_0(X+1)$ and $\mu_0(kN)=1/k$.

Let $\mu_n(X)=\frac{1}{n}\sum_{k=0}^{n-1} p^k\mu_0(p^kX)$ and $\eta_p(X)=\lim_\nu \mu_n(X)$. It is easily checked that η_p is a finitely additive measure and $\eta_p(X)=\eta_p(X+1)$. We will show

5) $\eta_p(pX)=\frac{1}{p}\eta_p(X)$.

For each $n\geqq 1$, we have

$$6)\quad |\mu_n(X)-p\mu_n(pX)|=\left|\frac{1}{n}\sum_{k=0}^{n-1} p^k\mu_0(p^kX)-\frac{p}{n}\sum_{k=0}^{n-1} p^k\mu_0(p^{k+1}X)\right| =\frac{1}{n}|\mu_0(X)-p^n\mu_0(p^nX)|\leqq\frac{1}{n},$$

because $\mu_0(X)\leqq 1$ and $\mu_0(p^nX)\leqq\mu_0(p^nN)=1/p^n$. Applying $\lim_\nu$ to 6), we get 5).

We define $F_p(m)=k$ where $m=p^i(kp-j)$ for some i and $1\leqq j<p$. For any $i=0,1,2,\cdots$ and $j=2,3,4,\cdots,p-1$, let $S_j^i=\{p^i(kp-j)|k=1,2,3,\cdots\}$, $S_j=\bigcup_{i=0} S_j^i$, $T^i=\{p^i(kp-1)|k=1,2,3,\cdots\}$, and $T=\bigcup_{i=0} T^i$.

Define a function $G:\bigcup_{2\leqq j<p} S_j\to T$ as $G(p^i(kp-j))=p^i(kp-1)$.

Since $T^0, T^0-1,\cdots,T^0-p+1$ are mutually disjoint and their union is N, $\eta_p(T^0)=1/p$. Therefore $\eta_p(S_j^i)=\eta_p(T^i-p^i(j-1))=\eta_p(T^i)=\eta_p(p^iT^0)$ $=1/p^{i+1}$. We show $\eta_p(S_j)=\eta_p(T)=1/(p-1)$. For S_j^i, T^i are mutually disjoint and $\bigcup_{i=0}^{n} T^i \subset T \subset N - \bigcup_{j=2}^{p-1}\bigcup_{i=0}^{n} S_j^i$ then $\sum_{i=0}^{n} \frac{1}{p^{i+1}} \leqq \eta_p(T) \leqq 1-(p-2) \times \sum_{i=0}^{n} \frac{1}{p^{i+1}}$. Let $p \to \infty$, we have $\eta_p(T) = 1/(p-1)$. Similarly $\eta_p(S_j) = 1/(p-1)$.

Remark. $\eta_p(S_j) = \sum_{i=0}^{\infty} \eta_p(S_j^i)$ and $\eta_p(T) = \sum_{i=0}^{\infty} \eta_p(T^i)$.

Lemma 1. *Let η be a finitely additive measure on N and $A = \bigcup_{i=0}^{\infty} A_i$ (disjoint union). If $\eta(A) = \sum_{i=0}^{\infty} \eta(A_i)$, then for any $X \subseteq N$, $\eta(X \cap A) = \sum_{i=0}^{\infty} \eta(X \cap A_i)$.*

Proof. Since A_i are mutually disjoint and

$$\bigcup_{i=0}^{n} (X \cap A_i) \subset (X \cap A) \subset \left(\bigcup_{i=0}^{n} (X \cap A_i) \cup \bigcup_{i=n+1}^{\infty} A_i\right),$$

$$\sum_{i=0}^{n} \eta(X \cap A_i) \leqq \eta(X \cap A) \leqq \sum_{i=0}^{n} \eta(X \cap A_i) + \sum_{i=n+1}^{\infty} \eta(A_i).$$

By letting $n \to \infty$, Lemma 1 follows because $\sum_{i=n+1}^{\infty} \eta(A_i)$ tends to 0.

Now we prove

7) $F_p^*(\eta_p) = \eta_p$.

We will show $\eta_p(X) = \eta_p(F_p^{-1}(X))$ for any $X \subseteqq N$. Let $A_n = T^n \cup \bigcup_{j=2}^{p-1} S_j^n$ and $B_n = \bigcup_{k=0}^{n} A_k$. The sets A_n are pairwise disjoint and $\eta_p(A_n) = (p-1)/p^{n-1}$, $\eta_p(B_n) = 1 - 1/p^{n-1}$. It follows from the definition of F_p that for each $n \in N$, $F_p^{-1}(X) \cap A_n = \bigcup_{j=1}^{p-1} p^n(pX - j)$. Consequently, if we denote $a = \eta_p(X)$, then

$$\eta_p(F_p^{-1}(X) \cap B_n) = a\left(1 - \frac{1}{p^{n+1}}\right) \quad \text{and}$$

$$\eta_p(B_n - F_p^{-1}(X)) = (1-a)\left(1 - \frac{1}{p^{n+1}}\right).$$

Now if n tends to infinity, $\eta_p(F_p^{-1}(X)) = a$ which proves 7).

Next we show

8) if $X \subseteq N$ is such that F_p is one-to-one on X, then $\eta_p(X) \leqq 1/(p-1)$.

By Lemma 1 and Remark,

$$\begin{aligned}\eta_p(X \cap S_j) &= \sum_{i=0}^{\infty} \eta_p(X \cap S_j^i) = \sum_{i=0}^{\infty} \eta_p(X \cap S_j^i + (j-1)3^i) \\ &= \sum_{i=0}^{\infty} \eta_p(G(X \cap S_j^i)) = \sum_{i=0}^{\infty} \eta_p(G(X \cap S_j^i) \cap T_j) = \eta_p(G(X \cap S_j)).\end{aligned}$$

Let $Y = (X \cap T) \cup \bigcup_{j=2}^{p-1} G(X \cap S_j)$. Since F_p is one-to-one on X, $X \cap T$ and $G(X \cap S_j)$ $(j = 2, 3, \cdots, p-1)$ are pairwise disjoint. Then $Y \subseteqq T$ and

$\eta_p(X)=\eta_p(Y)\leqq\eta_p(T)=1/(p-1)$.

Now by 7) and 8), Proposition 1 follows.

4. Proof of Proposition 2. Let us start with the proof of the following

Lemma 2. *Let* $N=\bigcup_{i=1}^{\infty}N_i=\bigcup_{j=1}^{\infty}M_j$ *(disjoint union), for all i and j* $|N_i|=|M_j|$, $1\in N_1\cap M_1$ *and for all* $n, n\in\bigcup_{i<n}N_i$ *and* $n\in\bigcup_{j<n}M_j$. *Then there exists a function* $f:N\xrightarrow[\text{onto}]{1;1}N$ *such that* $f(N_n)=M_{f(n)}$.

Proof. We define $f(i)$ for $i\in N_n$ by induction on n such that f is one-to-one and $f(N_n)=M_{f(n)}$.

We first put $f(1)=1$ and f to map N_1 one-to-one onto M_1. Then $f(N_1)=M_{f(1)}$ and f is one-to-one. If we define $f(i)$ for $i\in N_k$ $(k<n)$ such that $f(N_k)=M_{f(k)}$ and f is one-to-one on $\bigcup_{k<n}N_k$, then $f(n)$ is already defined because $n\in\bigcup_{k<n}N_k$. We take $f(i)$ for $i\in N_n$ such that f maps N_n one-to-one onto $M_{f(n)}$. Then $f(N_k)=M_{f(k)}$ for $k\leqq n$ and f is one-to-one on $\bigcup_{k\leqq n}N_k$.

We must prove f is onto. If not, we pick the least x such that $x\in N-f(N)$. Then for some $y<x$, $x\in M_y$. Since $y<x$, there is a z such that $f(z)=y$ and therefore $x\in M_y=f(N_z)$. So $x\in f(N)$. This contradiction proves Lemma 2.

Now we return to the proof of Proposition 2. Let $N_i=F_3^{-1}(i)$ and $M_j=F_p^{-1}(j)$. By Lemma 2, there is a function $f_p:N\xrightarrow[\text{onto}]{1;1}N$ such that $f_p(F_3^{-1}(i))=F_p^{-1}(f_p(i))$. So Proposition 2 holds.

References

[1] Zdenek Frolik: Fixed points of maps of βN. Bull. Amer. Math. Soc., **74**, 187–191 (1968).

[2] Thomas Jech and Karel Prickry: On projections of finitely additive measures (preprint).

[3] Mary Ellen Rudin: Partial orders on the types of βN. Trans. Amer. Math. Soc., **155**, 353–362 (1972).