2. *Studies on Holonomic Quantum Fields. XI*

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This paper is a direct continuation of our previous work [2]. We retain the same notations as in [2] without mentioning further.

1. In the present case of 2-dimensional Weyl equation, the orthogonal transformation $T[A]$ is the multiplication by $M(t)=M[A](t)$ where we have set $t=-x^-$. It is natural to ask if we can choose $Y_\pm$ and $Z_\pm$ to be multiplications by functions, say $Y_\pm(t)$ and $Z_\pm(t)$, respectively. The conditions (2) then require that $Y_+(t)$ and $Z_+(t)$ (resp. $Y_-(t)$ and $Z_-(t)$) are holomorphic in the upper (resp. the lower) half complex t-plane. This is the celebrated Riemann-Hilbert problem [1], [3].

Noting that $\lim_{|t|\to\infty} M(t)=1$, we can normalize $Y_\pm(t), Z_\pm(t)$ so that $\lim_{|t|\to\infty} Y_\pm(t)=\lim_{|t|\to\infty} Z_\pm(t)=1$. Then the unique solution is given by

$$X(t)=\sum_{n=0}^{\infty}(-)^n\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}dt_1\cdots dt_n c_n(t_1,\cdots,t_n;t)(M(t_1)-1)\cdots(M(t_n)-1), \tag{21}$$

$$c_n(t_1,\cdots,t_n;t)=\begin{cases}\dfrac{1}{2\pi}\dfrac{-i}{t_1-t_2-i0}\cdots\dfrac{1}{2\pi}\dfrac{-i}{t_n-t-i0} & \text{for } X=Y_+,\\ \dfrac{1}{2\pi}\dfrac{-i}{t-t_1-i0}\cdots\dfrac{1}{2\pi}\dfrac{-i}{t_{n-1}-t_n-i0} & \text{for } X=Y_-^{-1},\\ \dfrac{1}{2\pi}\dfrac{i}{t-t_1+i0}\cdots\dfrac{1}{2\pi}\dfrac{i}{t_{n-1}-t_n+i0} & \text{for } X=Z_+^{-1},\\ \dfrac{1}{2\pi}\dfrac{i}{t_1-t_2+i0}\cdots\dfrac{1}{2\pi}\dfrac{i}{t_n-t+i0} & \text{for } X=Z_-.\end{cases}$$

The kernel $\Phi(t,t')$ of $\Phi[T]$ in (3) reduces to

$$\Phi(t,t')=\frac{1}{2\pi i}\frac{1}{t-t'}(Y_-(t)^{-1}Y_+(t')-Z_+(t)^{-1}Z_-(t')). \tag{22}$$

In particular, we have

$$\begin{aligned}\Phi(t,t)&=\frac{1}{2\pi i}\left(\frac{dY_-(t)^{-1}}{dt}Y_+(t)-\frac{dZ_+(t)^{-1}}{dt}Z_-(t)\right),\\ &=\frac{-1}{2\pi i}\left(Y_-(t)^{-1}\frac{dY_+(t)}{dt}-Z_+(t)^{-1}\frac{dZ_-(t)}{dt}\right).\end{aligned} \tag{23}$$

Then from (7) we have the following

Theorem 4. *$\tau[T]$ is characterized by*

(24) $$2\delta \log \tau[T]=\int_{-\infty}^{\infty} dt \,\mathrm{trace}\, \delta M(t)\cdot \Phi(t,t)$$

and $\log \tau[1]=0$.

Corollary 4.1. *If $M(t)$ is abelian, i.e.* $[M(t), M(t')]=0$, *we have*

(25) $$2\log \tau[T]=-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dt}{2\pi}\frac{dt'}{2\pi}P\frac{1}{(t-t')^2}\,\mathrm{trace}\log M(t)\log M(t')$$

where P means the principal value.

Corollary 4.2. *If $A(x)$ is abelian, i.e.* $[A(x), A(x')]=0$, *we have*

(26) $$\log M(t)=\int_{-\infty}^{\infty} dx^+ A(t,x^+),$$

(27) $$2\log\tau[A]=-\int\int d^2x d^2x' P\frac{1}{(-x^-+x'^-)^2}\,\mathrm{trace}\, A(x)A(x').$$

Consider the limiting case where $M(t)$ is given by

(28) $$\frac{dM(t)}{dt}M(t)^{-1}=-2\pi i\sum_{\nu=1}^{n}L_\nu\delta(t-a_\nu),\qquad {}^tL_\nu=-L_\nu\,;$$

that is, $M(t)=M_\nu M_{\nu+1}\cdots M_n$ $(a_{\nu-1}<t<a_\nu,\ \nu=1,\cdots,n+1\,;\ a_0=-\infty,\ a_{n+1}=+\infty)$ with $M_\nu=\exp(2\pi i L_\nu)={}^tM_\nu^{-1}$. Here we assume $M_\infty=(M_1\cdots M_n)^{-1}=1$. Then $Y_\pm(t)$, $Z_\pm(t)$ are solutions of differential equations of the form [3] $\frac{dY_\pm}{dt}=\left(\sum_{\nu=1}^{n}\frac{A_\nu}{t-a_\nu}\right)Y_\pm$, $\frac{dZ_\pm}{dt}=\left(\sum_{\nu=1}^{n}\frac{B_\nu}{t-a_\nu}\right)Z_\pm$. If we denote by d the exterior differentiation with respect to $a_1,\cdots,a_n$, formula (24) gives

(29) $$\begin{aligned}2d\log\tau[T]&=\int_{-\infty}^{+\infty}\frac{dt}{2\pi i}\,\mathrm{trace}\Big\{dM(t)\cdot M(t)^{-1}\\&\quad\times\Big(-\lim_{\varepsilon(t)\downarrow 0}Y_+(t+i\varepsilon(t))^{-1}\frac{dY_+}{dt}(t+i\varepsilon(t))\\&\quad+\lim_{\eta(t)\downarrow 0}Z_-(t-i\eta(t))^{-1}\frac{dZ_-}{dt}(t-i\eta(t))\Big)\Big\}\\&=\frac{1}{2}\sum_{\mu\neq\nu}\mathrm{trace}\,A_\mu A_\nu\frac{da_\mu-da_\nu}{a_\mu-a_\nu}\\&\quad+\sum_{\nu=1}^{n}da_\nu\lim_{\varepsilon(a_\nu)\downarrow 0}\mathrm{trace}\,L_\nu^2\cdot\frac{1}{i\varepsilon(a_\nu)}\\&\quad+\frac{1}{2}\sum_{\mu\neq\nu}\mathrm{trace}\,B_\mu B_\nu\frac{da_\mu-da_\nu}{a_\mu-a_\nu}\\&\quad+\sum_{\nu=1}^{n}da_\nu\lim_{\eta(a_\nu)\downarrow 0}\mathrm{trace}\,L_\nu^2\cdot\frac{1}{-i\eta(a_\nu)}.\end{aligned}$$

After subtracting the normalization terms (the second and fourth terms), we thus obtain Theorem 2.4.7 in [3] (see also [1], [4]) for the operator $\varphi\otimes\varphi^{-1}$, $\varphi=\varphi(a_1\,;L_1)\cdots\varphi(a_n\,;L_n)$. This subtraction is unnecessary if we choose $\varepsilon(t)=\eta(t)$. Note that each of Y_+- and Z_--terms in (29) produces $d\log\tau$ for φ and φ^{-1} respectively.

2. Let $M(t)=\sum_{\nu=-\infty}^{\infty}M_\nu t^\nu$ be an $O(m)$-valued real analytic function defined on $S^1=\{t \in \boldsymbol{C} \mid |t|=1\}$. If we start from $S^1 \times \boldsymbol{R}=\{(t, x^+) \mid t \in S^1, x^+ \in \boldsymbol{R}\}$ instead of X^{Min} in § 2, we obtain analogous results, in particular, for the functional $\tau[T_M]$ where T_M means the multiplication by $M(t)$. The kernel functions of E_+ and E_- are given by

$$E_+(t, t')dt' = \underset{|t'|>|t|}{b.v.} \frac{1}{t'-t} \frac{dt'}{2\pi i} = \sum_{n\geqq 0} \left(\frac{t}{t'}\right)^n \frac{dt'}{2\pi i t'}, \tag{30}$$

$$E_-(t, t')dt' = \underset{|t|>|t'|}{b.v.} \frac{1}{t-t'} \frac{dt'}{2\pi i} = \sum_{n\geqq 1} \left(\frac{t'}{t}\right)^n \frac{dt'}{2\pi i t'}, \tag{31}$$

where $b.v.$ signifies the boundary value. Roughly speaking, $\det(E_+ + E_- T_M)$ means the determinant of the following matrix of infinite size (the left hand side of (32)):

$$\begin{bmatrix} M_0 & M_{-1} & M_{-2} & \cdots \\ M_1 & M_0 & M_{-1} & \cdots \\ M_2 & M_1 & M_0 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}, \qquad M^{(N)} = \begin{bmatrix} M_0 & M_{-1} & \cdots & M_{-N+1} \\ M_1 & M_0 & \cdots & M_{-N+2} \\ \vdots & \vdots & & \vdots \\ M_{N-1} & M_{N-2} & \cdots & M_0 \end{bmatrix}. \tag{32}$$

The determinant of this type is called the Toeplitz determinant. In the sequel we shall give an explicit formula for $\lim_{N\to\infty} \det M^{(N)}$ where $M^{(N)}$ is given by the right hand side of (32).

Let $Y_+(t)$ and $Z_+(t)$ (resp. $Y_-(t)$ and $Z_-(t)$) be holomorphic and invertible functions defined on $\{t \in \boldsymbol{C} \mid |t| \leqq 1\}$ (resp. $\{t \in \boldsymbol{C} \mid |t| \geqq 1\}$), satisfying $M(t) = Y_+(t)^{-1} Y_-(t) = Z_-(t)^{-1} Z_+(t)$. Let $c[M]$ be the functional of M characterized by $c[1]=1$ and

$$\delta \log c[M] = \int \operatorname{trace} \delta M(t) \cdot M(t)^{-1} \frac{dt}{2\pi i t}. \tag{33}$$

Explicitly written down,

$$c[M] = \det Y_+(0)^{-1} \cdot \det Y_-(\infty) = \det Z_-(\infty)^{-1} \cdot \det Z_+(0). \tag{34}$$

Then the limit

$$\sigma[M] = \lim_{N\to\infty} c[M]^{-N} \det M^{(N)} \tag{35}$$

exists and satisfies

$$\sigma[M] = \sigma[M^{-1}]. \tag{36}$$

Theorem 5. *The Toeplitz determinant $\sigma[M]$ is given by*

$$\sigma[M]^2 = \tau[T_M]. \tag{37}$$

Hence $\sigma[M]$ is characterized by $\sigma[1]=1$ and

$$\delta \log \sigma[M] = -\oint_{|t|=1} \frac{dt}{2\pi i} \operatorname{trace} \delta M(t) \left(Y_-(t)^{-1} \frac{dY_+(t)}{dt} - Z_+(t)^{-1} \frac{dZ_-(t)}{dt} \right). \tag{38}$$

Set

$$\hat{\omega}_n(\nu_1, \cdots, \nu_n) = \max(0, \nu_1, \nu_1+\nu_2, \cdots, \nu_1+\cdots+\nu_n) - \min(0, \nu_1, \nu_1+\nu_2, \cdots, \nu_1+\cdots+\nu_n), \tag{39}$$

$$\omega_n(t_1, \cdots, t_n) = \{-\omega_n^{(+)}(t_1, \cdots, t_n; t) - \omega_n^{(-)}(t_1, \cdots, t_n; t) + \omega_n^{(0)}(t_1, \cdots, t_n; t)\}|_{t=t_1}, \tag{40}$$

(41) $$\omega_n^{(\pm)}(t_1, \cdots, t_n; t) = E_\pm(t_1, t_2) \cdots E_\pm(t_n, t) dt_1 \cdots dt_n,$$

(42) $$\omega_n^{(0)}(t_1, \cdots, t_n; t) = \delta(t_1, t_2) \cdots \delta(t_n, t) dt_1 \cdots dt_n.$$

Here we have set $\delta(t, t') = E_+(t, t') + E_-(t, t')$.

Making use of the infinite series for $Y_\pm(t)$ and $Z_\pm(t)$ analogous to (21) we obtain the following

Corollary 5.1.

(43) $$\sigma[M] = \sum_{n=1}^{\infty} \frac{(-)^n}{n} \sum_{\nu_1 + \cdots + \nu_n = 0} \hat{\omega}_n(\nu_1, \cdots, \nu_n) \operatorname{trace} (M_{\nu_1} - \delta_{\nu_1 0}) \times \cdots (M_{\nu_n} - \delta_{\nu_n 0}),$$
$$= \sum_{n=1}^{\infty} \frac{(-)^n}{n} \oint_{|t_1|=1} \cdots \oint_{|t_n|=1} \omega_n(t_1, \cdots, t_n) \operatorname{trace} (M(t_1) - 1) \times \cdots (M(t_n) - 1).$$

Corollary 5.2. *If $M(t)$ is abelian, we have*

(44) $$\log \sigma[M] = - \oint_{|t|=1} \operatorname{trace} \log Y_-(t) \frac{d}{dt} \log Y_+(t) \frac{dt}{2\pi i}.$$

Remark 1. (38)–(44) are valid without the assumption that ${}^tM(t)^{-1} = M(t)$.

Remark 2. In the abelian case, if we set $\log M(t) = \sum_{n=-\infty}^{\infty} K_n t^n$, we have $\log c[M] = \operatorname{trace} K_0$ and $\log \sigma[M] = \operatorname{trace} \sum_{n=1}^{\infty} n K_n K_{-n}$. This is the well-known Szegö's theorem [5].

References

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