11. *A Remark on Convergence of Nonlinear Semigroups*

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1. Introduction. Let X be a real Banach space. Let A_n, $n=1, 2, \cdots$, and A be *dissipative operators* in X which satisfy the conditions

$$R(I-\lambda A_n)\supset\overline{D(A_n)} \quad \text{and} \quad R(I-\lambda A)\supset\overline{D(A)} \qquad \text{for } \lambda>0.$$

Let $\{T_n(t)\,;\,t\geqq 0\}$ and $\{T(t)\,;\,t\geqq 0\}$ be the (*nonlinear*) *semigroups* generated by A_n and A in the sense of Crandall-Liggett [6]. It was shown by Brezis-Pazy [4] that *if* $\overline{D(A)}\subset\overline{D(A_n)}$, $n=1,2,\cdots$, *then the following property* (i) *implies the property* (ii).

(i) $$\lim_{n\to\infty}(I-\lambda A_n)^{-1}=(I-\lambda A)^{-1}$$

for each $\lambda>0$ *and* $x\in\overline{D(A)}$.

(ii) $$\lim_{n\to\infty}T_n(t)=T(t)x$$

for each $x\in\overline{D(A)}$ *and the limit is uniform on bounded t-intervals.*

Our aim in this note is to show that the property (ii) implies (i) under some additional conditions. Precisely, we shall show the following

Theorem. *Let* X^* *be uniformly convex. If* $\overline{D(A)}$ *is convex and* $\overline{D(A)}\subset\overline{D(A_n)}$, $n=1,2,\cdots$, *then the property* (ii) *implies the property* (i).

The above theorem is due to Bénilan [3] in the Hilbert space case. The idea of our proof of the theorem is essentially due to the recent work [1] of Baillon. As usual, we define the duality map F on X into X^* by $F(x)=\{x^*\in X^*\,;\,\langle x,x^*\rangle=\|x\|^2=\|x^*\|^2\}$. If X^* is uniformly convex, then F is single-valued and uniformly continuous on each bounded set of X. We refer to Barbu [2] for some properties of the duality map and nonlinear semigroups.

2. Proof of Theorem. Let $\overline{D(A)}$ be convex and $\overline{D(A)}\subset\overline{D(A_n)}$, $n=1,2,\cdots$, and assume the property (ii). Let $x\in\overline{D(A)}$ and $\lambda>0$ be fixed. We set $y_n=(I-\lambda A_n)^{-1}x$. We want to show that y_n converges to $(I-\lambda A_n)^{-1}x$ as $n\to\infty$. For the purpose, we prepare some lemmas.

Lemma 1. $\|y_n\|=O(1)$ *as* $n\to\infty$.

Proof. By Theorem 9 in [4], we have

$$\|y_n-x\|\leqq\frac{4}{\lambda}\int_0^{\lambda}\|T_n(\tau)x-x\|\,d\tau.$$

Since $T_n(\tau)x$ is bounded as $n\to\infty$ uniformly for $\tau\in[0,\lambda]$ by (ii), it follows that $\|y_n\|$ is bounded as $n\to\infty$. Q.E.D.

By the Hahn-Banach theorem, there exists a linear functional L

on l^∞ such that

$$\underline{\lim}_{k\to\infty}\,\xi_k \leqq L(\{\xi_k\}) \leqq \overline{\lim}_{k\to\infty}\,\xi_k$$

for each $\{\xi_k\}\in l^\infty$. We choose such a functional L and define $\mathrm{LIM}_{k\to\infty}$ by $\mathrm{LIM}_{k\to\infty}\,\xi_k = L(\{\xi_k\})$. Apparently, $\mathrm{LIM}_{k\to\infty}$ is a bounded functional on l^∞ and enjoys the property that $\mathrm{LIM}_{k\to\infty}\,\xi_k \geqq 0$ if $\xi_k \geqq 0$. (See [7], p. 104.)

Let $\{y_{n(k)}\}$ be a subsequence of $\{y_n\}$ and define

$$\phi(y) = \frac{1}{2}\,\mathrm{LIM}_{k\to\infty}\,\|y_{n(k)} - y\|^2$$

for each $y \in \overline{D(A)}$. The functional ϕ is convex, continuous, bounded below and coercive (i.e., $\phi(y) \to +\infty$ as $\|y\| \to +\infty$). Since X is reflexive and $\overline{D(A)}$ is convex, we have the following (see [2], p. 52)

Lemma 2. *There exists a $y_0 \in \overline{D(A)}$ such that $\phi(y_0) = \inf\{\phi(y)\,;\, y \in \overline{D(A)}\}$.*

Let such a $y_0 \in \overline{D(A)}$ be fixed.

Lemma 3. $\mathrm{LIM}_{k\to\infty}\langle y_0 - y, F(y_0 - y_{n(k)})\rangle \leqq 0$ *for each* $y \in \overline{D(A)}$.

Proof. We follow the argument in [1]. Let $y \in \overline{D(A)}$ and $\varepsilon \in (0,1)$. It follows by a property of F (see [2]) that

$$\begin{aligned}&\langle y_0 - y, F(y_0 - y_n - \varepsilon(y_0 - y))\rangle\\ &\qquad \leqq (2\varepsilon)^{-1}(\|y_0 - y_n\|^2 - \|y_0 - y_n - \varepsilon(y_0 - y)\|^2).\end{aligned}$$

Let $n = n(k)$ and let $k \to \infty$. Then

$$\begin{aligned}&\mathrm{LIM}_{k\to\infty}\langle y_0 - y, F(y_0 - y_{n(k)} - \varepsilon(y_0 - y))\rangle\\ &\qquad \leqq \varepsilon^{-1}(\phi(y_0) - \phi((1-\varepsilon)y_0 + \varepsilon y)) \leqq 0.\end{aligned}$$

By letting $\varepsilon \to 0+$, we have the desired result, since F is uniformly continuous on bounded sets. Q.E.D.

Lemma 4. $\underline{\lim}_{k\to\infty}\|y_{n(k)} - y_0\| = 0$.

Proof. Since $u(t) = T_n(t)z$ is an integral solution of $u'(t) \in A_n u(t)$ for each $z \in \overline{D(A)}$ and $\lambda^{-1}(y_n - x) \in A_n y_n$, we have

$$\begin{aligned}&\frac{1}{2}\|T_n(t)z - y_n\|^2 - \frac{1}{2}\|z - y_n\|^2\\ &\qquad \leqq \int_0^t \langle \lambda^{-1}(y_n - x), F(T_n(\tau)z - y_n)\rangle d\tau\end{aligned} \tag{1}$$

for each $z \in \overline{D(A)}$ and $t \geqq 0$. (See [2].) Put $z = y_0$ and $n = n(k)$ in (1) and let $k \to \infty$. Then it follows by the uniform continuity of F that

$$\begin{aligned}0 &\leqq \phi(T(t)y_0) - \phi(y_0)\\ &\leqq \mathrm{LIM}_{k\to\infty}\int_0^t \langle \lambda^{-1}(y_{n(k)} - x), F(T(\tau)y_0 - y_{n(k)})\rangle d\tau.\end{aligned}$$

Divide this by $t > 0$ and let $t \to 0+$. Then it follows by the uniform continuity of F and Lemma 3 with $y = x$ that

$$\begin{aligned}0 &\leqq \mathrm{LIM}_{k\to\infty}\langle \lambda^{-1}(y_{n(k)} - x), F(y_0 - y_{n(k)})\rangle\\ &\leqq -\lambda^{-1}\,\mathrm{LIM}_{k\to\infty}\|y_0 - y_{n(k)}\|^2.\end{aligned}$$

Therefore, we obtain

$$0 \leqq \underline{\lim}_{k\to\infty} \|y_{n(k)} - y_0\|^2 \leqq \mathrm{LIM}_{k\to\infty} \|y_{n(k)} - y_0\|^2 \leqq 0.$$

Q.E.D.

Set $A_t = t^{-1}(T(t) - I)$ for $t > 0$. Since $\overline{D(A)}$ is convex and $x \in \overline{D(A)}$, there exists $(I - \lambda A_t)^{-1}x$ for $t > 0$.

Lemma 5. $\lim_{t\to 0+} (I - \lambda A_t)^{-1}x = (I - \lambda A)^{-1}x$.

Proof. Since $u(t) = T(t)z$ is an integral solution of $u'(t) \in Au(t)$ and $\lambda^{-1}((I - \lambda A)^{-1}x - x) \in A(I - \lambda A)^{-1}x$, we have

$$\begin{aligned} &\frac{1}{2}\|T(t)z - (I - \lambda A)^{-1}x\|^2 - \frac{1}{2}\|z - (I - \lambda A)^{-1}x\|^2 \\ &\qquad \leqq \int_0^t \langle \lambda^{-1}((I - \lambda A)^{-1}x - x), F(T(\tau)z - (I - \lambda A)^{-1}x)\rangle d\tau \end{aligned} \tag{2}$$

for each $z \in \overline{D(A)}$ and $t \geqq 0$. Put $z_t = (I - \lambda A_t)^{-1}x$ and let $z = z_t$ in (2). By using the fact that $t^{-1}(T(t)z_t - z_t) = \lambda^{-1}(z_t - x)$, we find easily that

$$\begin{aligned} &\langle z_t - x, F(z_t - (I - \lambda A)^{-1}x)\rangle \\ &\qquad \leqq \frac{1}{t}\int_0^t \langle (I - \lambda A)^{-1}x - x, F(T(\tau)z_t - (I - \lambda A)^{-1}x)\rangle d\tau, \end{aligned} \tag{3}$$

for $t > 0$. By Proposition 1 in [1], there exists $z_0 = \lim_{t\to 0+} z_t$. Therefore, by letting $t \to 0+$ in (3), we have

$$\begin{aligned} &\langle z_0 - x, F(z_0 - (I - \lambda A)^{-1}x)\rangle \\ &\qquad \leqq \langle (I - \lambda A)^{-1}x - x, F(z_0 - (I - \lambda A)^{-1}x)\rangle, \end{aligned}$$

which yields $z_0 = (I - \lambda A)^{-1}x$. Hence $\lim_{t\to 0+} z_t = (I - \lambda A)^{-1}x$. Q.E.D.

We have all the material to complete the proof of the theorem. Lemma 4 implies that there exists a subsequence $\{y_{n(k(j))}\}$ of $\{y_{n(k)}\}$ such that $\lim_{j\to\infty} y_{n(k(j))} = y_0$. Put $n = n(k(j))$ in (1) and let $j \to \infty$. Then we get just the same inequality as in (2) with $(I - \lambda A)^{-1}x$ replaced by y_0, for each $z \in \overline{D(A)}$ and $t \geqq 0$. Therefore, the same argument as in the proof of Lemma 5 implies also that $\lim_{t\to 0+}(I - \lambda A_t)^{-1}x = y_0$. So, by Lemma 5, it turns out that $y_0 = (I - \lambda A)^{-1}x$. Hence, $\lim_{j\to\infty} y_{n(k(j))} = (I - \lambda A)^{-1}x$ and $\lim_{n\to\infty} y_n = (I - \lambda A)^{-1}x$ as desired.

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