92. *The Invertibility Problem on Amphicheiral Excellent Knots*[*)]

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The invertibility problem of knots is an old problem in knot theory. Although specific examples of non-invertible knots are obtained by H. F. Trotter [12] and W. Whitten [14], any reasonable invertibility invariants for testing examples are not known. Following R. Riley [8], we call a tame knot k in a 3-sphere S^3 *excellent* when S^3-k has a hyperbolic structure, i.e., a complete Riemannian metric of constant negative curvature with finite volume. By Thurston's existence theorem [11] of a hyperbolic structure, we see that many knots are excellent. In this paper we shall present an invertibility invariant for amphicheiral excellent knots. This invariant is enough to make a complete list of prime knots up to 10 crossings which are non-invertible and amphicheiral. Let $\langle t\rangle$ be an infinite cyclic group with a generator t and $Z\langle t\rangle$ be its group ring. Let f_1 and f_2 be in $Z\langle t\rangle$. By $f_1\doteq f_2$ (or $f_1\doteq_2 f_2$) we mean that f_1 and f_2 (or the Z_2-reductions of f_1 and f_2) are equal up to units of $Z\langle t\rangle$ (or $Z_2\langle t\rangle$). Let $k(t)$ be the Alexander polynomial ($\in Z\langle t\rangle$) of a tame knot k in S^3. Let $p_\lambda(t)=(t^\lambda-1)/(t-1)$ for any integer $\lambda>0$.

Theorem 1. *Let k be an excellent knot. If k is negative-amphicheiral, then* (1) *$k(t^2)\doteq f(t)f(-t)$ for $f(t)\in Z\langle t\rangle$ with $f(-t)\doteq f(t^{-1})$ and $|f(1)|=1$. If k is positive-amphicheiral, then* (2) *either $k(t)\doteq f(t)^2$ for $f(t)\in Z\langle t\rangle$ with $f(t)\doteq f(t^{-1})$ and $|f(1)|=1$, or there exist positive integers n, λ with λ odd such that $k(t)\doteq f(t)^2 f_0(t)f_1(t)\cdots f_{n-1}(t)$ for $f(t), f_i(t)\in Z\langle t\rangle$ with $f(t)\doteq f(t^{-1})$, $f_i(t)\doteq f_i(t^{-1})$, $|f(1)|=|f_i(1)|=1$ and $f_i(t)\doteq_2 f(t)^{2^i+1}p_\lambda(t)^{2^i}$, $i=0,1,\cdots,n-1$. If k is invertible and amphicheiral, then $k(t)$ satisfies both* (1) *and* (2).

Let h denote a piecewise-linear auto-homeomorphism of S^3 with $h(k)=k$. Then k is (*periodically* or *strongly*, resp.) *amphicheiral* if there is an orientation-reversing h (or finite order or of order 2, resp.); more precisely, k is (*periodically* or *strongly*, resp.) *positive*- or *negative-amphicheiral* according to whether $h|k$ is orientation-preserving or -reversing. k is (*strongly*) *invertible* if there is an orientation-preserving h (of order 2) such that $h|k$ is orientation-reversing.

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Lemma 1. *An excellent knot k is periodically positive-amphicheiral, strongly negative-amphicheiral or strongly invertible, respectively, if it is positive-amphicheiral, negative-amphicheiral or invertible.*

Proof. Let h be an auto-homeomorphism of S^3 giving a positive-, negative-amphicheirality or invertibility of k. By Mostow's rigidity theorem [6] and L. C. Siebenmann [10, § 7, Assertion], after a homotopic deformation of h we can assume that h has a finite order. Then we may assume that h has order 2^{n+1}, $n \geq 0$. (If h has order $2^{n+1}m$, m odd, we can replace h by h^m, where note that the order of h is always even since h or $h|k$ is orientation-reversing.) To complete the proof, it suffices to prove that if $h|k$ is orientation-reversing and $n \geq 1$, then k is a trivial knot. This follows from F. Waldhausen [14] since k is the fixed point set of the orientation-preserving involution h^{2^n} by Smith theory [1]. This completes the proof.

Proof of Theorem 1. By Lemma 1 k is strongly negative-amphicheiral, if it is negative-amphicheiral. So, (1) follows from R. Hartley and the author [4]. Let k be positive-amphicheiral and hence periodically positive-amphicheiral by Lemma 1. Let h be an auto-homeomorphism of S^3 of order 2^{n+1}, $n \geq 0$, giving this amphicheirality of k. If $n=0$, k is strongly positive-amphicheiral, so by [4] $k(t) \doteq f(t)^2$ for $f(t) \in Z\langle t \rangle$ with $f(t) \doteq f(t^{-1})$ and $|f(1)|=1$. Let $n \geq 1$. h is orientation-reversing, so that $\mathrm{Fix}(h) \neq \phi$ by Lefschetz fixed point theorem. Hence $\mathrm{Fix}(h^2)$ is a knot, k^0, by Smith theory [1]. By [13] k^0 is a trivial knot, so that the orbit space $S_* = S^3/h^2$ is a 3-sphere. $k \cap k^0 = \phi$ since $h|k$ is orientation-preserving. It follows that k is a lift of some knot $k_* \subset S_*$ under the canonical 2^n-fold cyclic branched covering $S^3 \to S_*$ branched along some trivial knot $k_*^0 \subset S_*$. Since k is connected, the linking number, λ, of k_* and k_*^0 must be odd. Orient k_*^0 so that $\lambda > 0$. Let $d(t_1, t_2)$ be the (integral) Alexander polynomial of the link $k_* \cup k_*^0 \subset S_*$. Define $d_i(t) = \prod_{\omega_i} d(t, \omega_i)$ ($\in Z\langle t \rangle$) where ω_i ranges over all 2^i-th roots of unity, and $f_i(t) = d_{i+1}(t)/d_i(t)$ ($\in Z\langle t \rangle$), $i=0,1,2,\cdots$. Since $d(t_1, t_2) = \pm t_1^a t_2^b d(t_1^{-1}, t_2^{-1})$ ($a, b \in Z$), we see that $d_i(t) \doteq d_i(t^{-1})$ and hence $f_i(t) \doteq f_i(t^{-1})$. By K. Murasugi [7, Theorem 1 and Propositions 4.1, 4.2], $p_\lambda(t)k(t) \doteq d_n(t) = d_0(t)f_0(t)\cdots f_{n-1}(t)$, $d_0(t) = d(t,1) \doteq k_*(t)p_\lambda(t)$ and $d_i(t) \doteq_2 d(t,1)^{2^i} \doteq k_*(t)^{2^i} p_\lambda(t)^{2^i}$. It follows that $k(t) \doteq k_*(t)f_0(t)\cdots f_{n-1}(t)$ and $f_i(t) \doteq_2 k_*(t)^{2^i} p_\lambda(t)^{2^i}$. $|f_i(1)|=1$ follows from $|k(1)|=1$. To complete the proof, it suffices to prove that $k_*(t) \doteq f(t)^2$ for $f(t) \in Z\langle t \rangle$ with $f(t^{-1}) \doteq f(t)$ and $|f(1)|=1$. This follows from [4], since an involution of S_* induced by h gives a strong positive-amphicheirality of k_*. This completes the proof.

The following is a revised special case of Thurston's existence theorem of a hyperbolic structure [11] and due to R. Riley [8, Corollary to Theorem 1].

Lemma 2. *A non-trivial,* 2*-bridged or prime* 3*-bridged knot is excellent if and only if it is not a torus knot.*

Corollary. (1) *Any* 2*-bridged or prime invertible* 3*-bridged knot is strongly invertible.* (2) *A* 2*-bridged or prime* 3*-bridged knot is excellent if it is amphicheiral.*

Proof. Any 2-bridged knot is invertible. (1) follows from Lemmas 1, 2 and the fact that a torus knot is strongly invertible. (Let $S^1 =\{z \in \boldsymbol{C} \,|\, |z|=1\}$ and $S^3=\{(z_1, z_2) \in \boldsymbol{C}^2 \,|\, |z_1|^2+|z_2|^2=2\}$. The (p, q)-torus knot $k(p, q)$ with p, q coprime is the image of the imbedding $S^1 \to S^1 \times S^1 \subset S^3$ sending z to (z^p, z^q). Then the complex conjugation gives a strong invertibility of $k(p, q)$.) (2) follows from Lemma 2 since no amphicheiral knot is a torus knot. (To see this, use a local knot signature argument (cf. J. W. Milnor [5]).)

The strong invertibility of a 2-bridged knot has been pointed out also by J. M. Montesinos. Also, in [4] we have known that any 2-bridged amphicheiral knot is strongly negative-amphicheiral, but not strongly positive-amphicheiral.

Here is a list of prime amphicheiral knots up to 10 crossings (in the notation of Rolfsen's book [9]). (Cf. J. H. Conway [3].) $4_1, 6_3, 8_3, 8_9, 8_{12}, 8_{17}, 8_{18}, 10_{17}, 10_{33}, 10_{37}, 10_{43}, 10_{45}, 10_{79}, 10_{81}, 10_{88}, 10_{99}, 10_{109}, 10_{115}, 10_{118}, 10_{123}$. Clearly, these are 2-bridged or 3-bridged. So, they are excellent by Corollary (2). Since they are negative-amphicheiral (cf. [3]), they are all strongly negative-amphicheiral by Lemma 1. This has been proved also by Van Buskirk [2]. The knots other than $8_{17}, 10_{79}, 10_{81}, 10_{88}, 10_{109}, 10_{115}, 10_{118}$ are known to be invertible (cf. [3]). So, by Lemma 1 they are strongly invertible and periodically positive-amphicheiral, among which the only strongly positive-amphicheiral knots are 10_{99} and 10_{123} ([4]).

Theorem 2. *The remaining knots* $8_{17}, 10_{79}, 10_{81}, 10_{88}, 10_{109}, 10_{115}, 10_{118}$ *are all non-invertible.*

The proof follows by checking that none of them satisfies the condition (2) of Theorem 1.

We note that the non-invertibility of 8_{17} has been proved also, by a geometric method, by F. Bonahon and L. C. Siebenmann in Low-Dimensional Topology Conference at Bangor, 1979.

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