# 91. *The Space of Distributions Treated as a Ranked Space*

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For the space $(\mathcal{D})$, K. Kunugi pointed out in [4] that, if we use the method of ranked spaces introduced by Kunugi in 1954 [3], $(\mathcal{D})$ can be treated as a space having a countable base (cf. [6, III, § 1]). In this paper we will show that the same assertion also holds for the space $(\mathcal{D}')$ of distributions. Indeed, the space $(\mathcal{D}')$ can be defined as a ranked space having a countable base, in such a way that the $r$-convergence in the ranked space $(\mathcal{D}')$ coincides with the weak convergence in $(\mathcal{D}')$ (cf. [6, III, § 3]). We moreover show that the ranked space $(\mathcal{D}')$ so defined is a complete ranked vector space satisfying the $r$-second countability axiom, and show that the family of $r$-Borel sets in the ranked space $(\mathcal{D}')$ coincides with the family of Borel sets in the weak topology of $(\mathcal{D}')$.

For notations and definitions in the distribution theory and the ranked space theory we refer to [6] and [5], respectively. In particular, we say that the base of a ranked space $E$ is countable if, for each $p \in E$ and for each $n \in N$, where $N=\{0, 1, 2, \cdots\}$, preneighborhoods of $p$ of rank $n$ are at most countable infinity; and say that a ranked space $E$ satisfies the $r$-second countability axiom if there exists a countable collection $\mathcal{W}$ of preneighborhoods such that, for any $r$-open set $O$ in $E$ and any point $p \in O$, there exists a $W \in \mathcal{W}$ such that $p \in W \subset O$. We call the members of the smallest $\sigma$-algebra which contains all of the $r$-open sets in a ranked space $E$ the $r$-Borel sets in $E$.

We first give the definition of the ranked space $(\mathcal{D})$ in a slight modification of the definition of Kunugi. For $l \in N$, by $(\mathcal{D}_l)$ we denote the vector subspace of $(\mathcal{D})$ consisting of all functions of $(\mathcal{D})$ which vanish outside the set $K(l)=\{x=(x_1, \cdots, x_n) \in \boldsymbol{R}^n : |x_i| \leqq l+1 \text{ for } i=1, \cdots, n\}$. Consider in $(\mathcal{D}_l)$ the countable system of norms:

$$\|\phi\|_m = \sup \{\sup_x |D^p\phi(x)| : p=(p_1, \cdots, p_n),\ p_i \leqq m \text{ for } i=1, \cdots, n\}, \qquad (m \in N).$$

Corresponding to $m \in N$ and $\varepsilon > 0$, consider the set $\{\phi \in (\mathcal{D}_l) : \|\phi\|_m < \varepsilon\}$, denoted by $S(l, m, \varepsilon)$. We define, for each $l \in N$, the ranked space $(\mathcal{D}_l)$ as a ranked space $((\mathcal{D}_l), \mathcal{V}^l(\phi), \mathcal{V}^l_m)$ provided with $\mathcal{V}^l(\phi) = \{\phi + S(l, m, \varepsilon) : m \in N, \varepsilon > 0\}$ and $\mathcal{V}^l_m = \{\phi + S(l, m, 1/2^m) : \phi \in (\mathcal{D}_l)\}$; and define the ranked space $(\mathcal{D})$ as a ranked space $((\mathcal{D}), \mathcal{V}(\phi), \mathcal{V}_m)$ provided with $\mathcal{V}(\phi) = \{\phi + S(l, m, \varepsilon): l, m \in N, \varepsilon > 0\}$ and $\mathcal{V}_m = \{\phi + S(l, m, 1/2^m): l \in N, \phi \in (\mathcal{D})\}$. We will denote the preneighborhood $\phi + S(l, m, 1/2^m)$ of rank $m$ of $\phi$

by $V(\phi; l, m)$.

**Lemma 1.** *For some $\alpha>0$, let $M\subset\{\phi\in(\mathcal{D}_l): \|\phi\|_{m+1}\leqq\alpha\}$. Then, for any $\varepsilon>0$, there exist $\phi_1, \cdots, \phi_s\in M$ such that $M\subset\bigcup_{i=1}^{s}\{\phi_i+S(l, m, \varepsilon)\}$* (cf., for example, [1, p. 55]).

**Proposition 1.** *For each ranked space $(\mathcal{D}_l)$ $(l\in N)$, there exists a countable set which is dense in the ranked space $(\mathcal{D}_l)$.*

A functional defined on $(\mathcal{D})$ is $r$-continuous in the ranked space $(\mathcal{D})$ if and only if it is $r$-continuous in every ranked space $(\mathcal{D}_l)$ $(l\in N)$; a functional defined on $(\mathcal{D}_l)$ is $r$-continuous in the ranked space $(\mathcal{D}_l)$ if and only if it is continuous in the topology of the ordinary sense defined by the countable system $\|\ \|_m$ $(m\in N)$ of norms (see [1, p. 19]); and a linear functional defined on $(\mathcal{D})$ is a distribution if and only if it is $r$-continuous in the ranked space $(\mathcal{D})$.

We denote the set $\{\phi\in(\mathcal{D}_l): \|\phi\|_m\leqq 1\}$ by $B_m^l$. For a linear functional $f$ defined on $(\mathcal{D})$, we define $\|f\|_m^l=\sup T_l$ if $T_l$ is bounded above, and $=+\infty$ if $T_l$ is unbounded above, where $T_l=\{|f(\phi)|; \phi\in B_m^l\}$.

**Lemma 2.** *Let*

$$A=\{f\in(\mathcal{D}'): \|f\|_{m_i}^i<\alpha_i \text{ for } i=0, 1, \cdots, j\},$$
$$B=\{f\in(\mathcal{D}'): \|f\|_{n_i}^i<\beta_i \text{ for } i=0, 1, \cdots, j'\}.$$

*Then, if $A\supset B$, $j\leqq j'$ and $n_0\leqq n_1\leqq\cdots\leqq n_{j'}$, it holds that $m_i\geqq n_i$ for $i=0, 1, \cdots, j$.*

**Lemma 3.** *Let $A$ and $B$ be as in Lemma 2. If $A\supset B$ and $n_0\leqq n_1\leqq\cdots\leqq n_{j'}$, then $j\leqq j'$.*

Lemma 2 can elementarily be proved, and Lemma 3 is immediate from Lemma 2.

Corresponding to a system of non-negative integers: $m_0\leqq m_1\leqq\cdots\leqq m_j$, consider the set

$$\{f\in(\mathcal{D}'): \|f\|_{m_i}^i<1/2^j \text{ for } i=0, 1, \cdots, j\},$$

denoted by $K(j, \{m_i\})$. Then

**Lemma 4.** *$K(j, \{m_i\})\supset K(j', \{m_i'\})$ holds if and only if $j\leqq j'$ and $m_i\geqq m_i'$ for $i=0, 1, \cdots, j$.*

**Proof.** The "if" part is immediate from the fact that $\|f\|_{m'}^l\geqq\|f\|_m^l$ if $m'\leqq m$. The "only if" part follows from Lemmas 2 and 3.

Let $f\in(\mathcal{D}')$. Corresponding to each $j\in N$ and each system of non-negative integers: $m_0\leqq m_1\leqq\cdots\leqq m_j$, we define a preneighborhood of $f$ by $f+K(j, \{m_i\})$. A preneighborhood $f+K(j, \{m_i\})$ is said to be of rank $j$. Denote by $\mathcal{V}(f)$ the family of all preneighborhoods of $f$ and by $\mathcal{V}_j$ the family of all preneighborhoods of rank $j$. Then, the space $(\mathcal{D}')$ provided with $\mathcal{V}(f)$ $(f\in(\mathcal{D}'))$ and $\mathcal{V}_j$ $(j\in N)$ becomes a ranked space. Moreover, as is easily seen, the base of the ranked space $(\mathcal{D}')$ is countable. From now on the ranked space $(\mathcal{D}')$ means the ranked space $(\mathcal{D}')$ so defined. We will denote the preneighborhood $f+K(j, \{m_i\})$ of rank $j$ of $f$ by $V(f; j, \{m_i\})$.

**Proposition 2.** (1) *The family of preneighborhoods in the ranked space* $(\mathscr{D}')$ *satisfies the axioms* (B) *and* (C) *of Hausdorff* (see [2, p. 213]).

(2) *The ranked space* $(\mathscr{D}')$ *is a ranked vector space satisfying the condition* (b) *of Proposition* 29 *in* [5].

**Lemma 5.** *Let* $V(f;j,\{m_i\})\supset V(f;j',\{m'_i\})\supset V(g;j'',\{n_i\})$ *and* $j<j'$. *Then,*

(1) $(j<)j'\leqq j''$, *and*

(2) $m_i\geqq m'_i$ *for* $i=0,1,\cdots,j$ *and* $m'_i\geqq n_i$ *for* $i=0,1,\cdots,j'$.

This can be proved by using Lemmas 2 and 3. Lemmas 4 and 5 play a central role in our methods.

**Lemma 6.** *Let* $\{f_j\}$ *be a Cauchy sequence in the ranked space* $(\mathscr{D}')$. *Then, for any* $\phi\in(\mathscr{D})$, $\{f_j(\phi)\}$ *is a Cauchy sequence in the complex number field.*

**Proof.** By the assumption, there exists a canonical fundamental sequence $u=\{V_i=V(g_i;k_i,\{m_t^i\})\}$ such that, for every $i\in N$, a $j_i$ can be found with the property that, if $j\geqq j_i$, then $f_j\in V_i$. Since $u$ is canonical, $i\leqq k_i$. Let $\phi(\neq 0)\in(\mathscr{D})$. Then, $\phi$ belongs to some $(\mathscr{D}_l)$. For each $i\geqq l$, consider the member $m_l^l$ of $\{m_t^l: t=0,1,\cdots,k_l\}$ and the member $m_l^i$ of $\{m_t^i: t=0,1,\cdots,k_i\}$. Then, by Lemma 5 and the fact that $u$ is canonical, $m_l^l\geqq m_l^i$ holds. Therefore, if we put $\psi=(1/\kappa)\phi$, where $\kappa=\|\phi\|_{m_l^l}$, then, $\|\psi\|_{m_l^i}\leqq\|\psi\|_{m_l^l}=1$. On the other hand; for $j',j''\geqq j_i$, $\|f_{j'}-f_{j''}\|^l_{m_l^i}<1/2^{k_i-1}$. Hence, $|f_{j'}(\phi)-f_{j''}(\phi)|<\kappa/2^{k_i-1}$.

**Theorem 1.** *Let* $f_j, f\in(\mathscr{D}')$ $(j=1,2,\cdots)$. *Then,* $\{f_j\}$ *r-converges to* $f$ *in the ranked space* $(\mathscr{D}')$ *if and only if* $\{f_j\}$ *converges weakly to* $f$.

**Proof.** The "if" part. For each $l\in N$, an $m_l\in N$ can be found with the property that, for each $i\in N$, there exists a $k_i^l$ such that, if $j\geqq k_i^l$, then $\|f-f_j\|^l_{m_l}<1/2^i$ (see [1, p. 57]). Then, we can choose $\{m_l\}$ in such a way that $m_l\leqq m_{l+1}$. For such a $\{m_l\}$, define the sequence of preneighborhoods $\{V_i=V(f;i,\{m_0\leqq\cdots\leqq m_i\})$. Then, the sequence is fundamental and $f_j\in V_i$ for every $j\geqq\max(k_i^0,k_i^1,\cdots,k_i^i)$. The "only if" part is immediate from Lemma 6.

**Theorem 2.** *The ranked space* $(\mathscr{D}')$ *is complete.*

**Proof.** For a fundamental sequence $\{V_i=V(g_i;k_i,\{m_j^i\})\}$, by Lemma 6 there exists $\lim g_i(\phi)$ for any $\phi\in(\mathscr{D})$. Set $g(\phi)=\lim g_i(\phi)$. Then $g\in(\mathscr{D}')$ (see [1, p. 68]). Moreover $g\in\bigcap V_i$.

**Lemma 7.** *Consider a system of non-negative integers*: $m_0\leqq m_1\leqq\cdots\leqq m_j$. *Let, for some* $\alpha>0$,

$$M\subset\{f\in(\mathscr{D}'):\|f\|^i_{m_i}\leqq\alpha \text{ for } i=0,1,\cdots,j\}.$$

*Then, there exist* $f_1,\cdots,f_p\in M$ *such that* $M\subset\bigcup_{k=1}^p V(f_k;j,\{m_i+1\})$.

**Proof.** Let $i\in\{0,1,\cdots,j\}$ and set $\varepsilon=1/(\alpha\cdot 2^{j+2})$. Then, by Lemma 1 there exist $\phi_1^i,\cdots,\phi_{t_i}^i\in B^i_{m_i+1}$ such that $B^i_{m_i+1}\subset\bigcup_{s=1}^{t_i}\{\phi_s^i+S(i,m_i,\varepsilon)\}$. We put $t=\sum t_i$, and we make correspond to each $f\in M$ a point $\nu(f)$ of $t$-dimensional complex Euclidean space defined by

$$\nu(f)=(f(\phi_1^0),\cdots,f(\phi_{t_0}^0),f(\phi_1^1),\cdots,f(\phi_{t_1}^1),\cdots,f(\phi_1^j),\cdots,f(\phi_{t_j}^j)).$$

Set $H=\{\nu(f): f\in M\}$. Then, by the assumption, there exists a finite covering of $H$ consisting of solid spheres $\{O_k: k=1,\cdots,p\}$ with $\mathrm{diam}(O_k)<1/2^{j+2}$ and such that $O_k\cap H\neq\emptyset$. The desired assertion is true for a system $f_k\in M$ $(k=1,\cdots,p)$ so chosen that $\nu(f_k)\in O_k$.

**Theorem 3.** *The ranked space $(\mathcal{D}')$ satisfies the r-second countability axiom.*

**Proof.** Denote a system of non-negative integers: $m_0\leqq m_1\leqq\cdots\leqq m_j$ by $\tau$. Corresponding to a $\tau$ and a $k\in N$, set $K_{\tau k}=\{f\in(\mathcal{D}'): \|f\|_{m_i}^i\leqq k+1$ for $i=0,1,\cdots,j\}$. For each $K_{\tau k}$, by Lemma 7 there exist $f_{\tau ks}\in K_{\tau k}$ $(s=1,\cdots,p_{\tau k})$ such that $K_{\tau k}\subset\bigcup_{s=1}^{p_{\tau k}}V(f_{\tau ks}; j,\{m_i+1\})$. The desired assertion is true for the countable collection consisting of all the preneighborhoods so chosen for all pairs $\tau, k$.

Denote by $\mathcal{B}_w$ the family of Borel sets in the weak topology of $(\mathcal{D}')$ and by $\mathcal{B}_R$ the family of $r$-Borel sets in the ranked space $(\mathcal{D}')$.

**Lemma 8.** *Every set which is open in the weak topology of $(\mathcal{D}')$ is also r-open in the ranked space $(\mathcal{D}')$.*

**Proof.** Consider a weak neighborhood of $f\in(\mathcal{D}'): W(f)=f+\{g\in(\mathcal{D}'): |g(\phi_t)|<\varepsilon$ for $t=1,\cdots,s\}$. For $r=f+g\in W(f)$, consider any canonical fundamental sequence $u=\{V_j=V(r; k_j,\{m_0^j\leqq\cdots\leqq m_{k_j}^j\})$ of center $r$. Take an $l\in N$ such that $(\mathcal{D}_l)\ni\phi_1,\cdots\phi_s$, and let $j$ be the smallest integer such that $l\leqq k_j$. Take a $j'$ such that $k_{j'}\geqq\max(l,(\lambda+1)/\alpha)$, where $\lambda=\max_t\|\phi_t\|_{m_l^j}$ and $\alpha=\min_t(\varepsilon-|g(\phi_t)|)$. Then, we have $V_{j'}\subset W(f)$.

**Lemma 9.** *Let $M=\{f\in(\mathcal{D}'): \|f\|_m^l\leqq\alpha\}$. Then, $M\in\mathcal{B}_w$.*

**Proof.** By Proposition 1, there exists a countable set $\{\phi_j\}$ which is dense in $B_m^l$. For each $k\in\{1,2,\cdots\}$, put $M_k=\{f\in(\mathcal{D}'): |f(\phi_j)|\leqq\alpha$ for $j=1,\cdots,k\}$. Then, $M=\bigcap_{k=1}^\infty M_k$.

From Lemmas 8 and 9 and Theorem 3, it follows that

**Theorem 4.** *$\mathcal{B}_R$ coincides with $\mathcal{B}_w$.*

## References

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