The purpose of this note is to prove the following

Theorem. The only integer solutions of the Diophantine equation

\[ 3y^2 = x^3 + 2x \]  

are given by \( x = 0, 1, 2 \) and \( 24 \).

By a classical theorem of A. Thue on the elliptic Diophantine equation we know that the equation (1) has only finitely many solutions in integers \( x \) and \( y \). In order to effectively determine all the solutions of (1), we shall make use of some results due to W. Ljunggren [1], [2], and [3].

We write the equation (1) in the form

\[ y^2 = \frac{1}{3} x(x^2 + 2) \]

and distinguish three cases according as \( x \equiv 0, 1 \) or \( 2 \) (mod 3).

Solutions with \( x \equiv 0 \) (mod 3). Write \( x = 3x_1 \). We have then \( y^2 = x_1 \cdot (9x_1^2 + 2) \), where \( d = \text{g.c.d.} \ (x_1, 9x_1^2 + 2) = 1 \) or \( 2 \).

If \( x_1 \) is an odd integer, then \( d = 1 \) and we have \( x_1 = Y^2, 9x_1^2 + 2 = X^2 \) for some integers \( X, Y \) with \( \text{g.c.d.} \ (X, Y) = 1 \). Eliminating \( x_1 \) from these equations, we get \( X^2 - 9Y^2 = 2 \); but this equation has no integer solutions \( X, Y \), since the congruence \( X^2 \equiv 2 \) (mod 3) is insoluble.

If \( x_1 \) is an even integer, then \( d = 2 \) and so \( x_1 = 2Y^2, 9x_1^2 + 2 = 2X^2 \) for some integers \( X, Y \) with \( \text{g.c.d.} \ (X, Y) = 1 \). Eliminating \( x_1 \), we get the equation

\[ X^2 - 18Y^4 = 1, \]

which can be rewritten in the form \( X^2 - 2(3Y^2)^2 = 1 \).

Now, the solutions in non-negative integers \( u, v \) of the equation

\[ u^2 - 2v^2 = 1 \]

are given by \( u = u_{2m}, v = v_{2m} \) \((m = 0, 1, 2, \ldots)\), where

\[ u_n + \sqrt{2} v_n = (1 + \sqrt{2})^n \quad (n = 0, 1, 2, \ldots). \]

The sequences \( u_n, v_n \) are determined by the relations

\[ u_0 = 1, \quad u_1 = 1, \quad u_{n+1} = 2u_n + u_{n-1} \quad (n \geq 1), \]
\[ v_0 = 0, \quad v_1 = 1, \quad v_{n+1} = 2v_n + v_{n-1} \quad (n \geq 1). \]

**Lemma 1.** We have for all \( m \geq 0 \)

In fact, the equation (1) arises from a problem concerning MacMahon's 'chromatic' triangles in graph theory and, according to M. Gardner, it is known that the only solutions of (1) with \( x \leq 5,000 \) are as listed in the theorem.
g.c.d. \((u_m, v_m) = \text{g.c.d.} \ (u_m, u_{2m}) = \text{g.c.d.} \ (u_{2m}, v_m) = 1.\)

Proof will be easily carried out by noticing the relations
\[ u_n^2 - 2v_n^2 = (-1)^n \quad (n \geq 0) \]
and
\[ u_{2n} = u_n^2 + 2v_n^2 \quad (n \geq 0) \]
which is a special case of
\[ u_{m+n} = u_m u_n + 2v_m v_n \quad (m, n \geq 0). \]

**Lemma 2.** We have
\[ u_n \equiv 0 \pmod{3} \quad \text{if and only if } n \equiv 2 \pmod{4} \]
and
\[ v_n \equiv 0 \pmod{3} \quad \text{if and only if } n \equiv 0 \pmod{4}. \]

**Proof.** Indeed, we observe that
\[
\begin{array}{cccccccc}
 n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
u_n & 1 & 1 & 0 & 1 & 2 & 2 & 0 & 2 \\
v_n & 0 & 1 & 2 & 2 & 0 & 2 & 1 & 1 \\
\end{array}
\]
(mod 3)

This can be readily verified by making use of the defining relations for \(u_n\) and \(v_n\), or of the relations (5) and
\[ v_{m+n} = u_m v_n + u_n v_m \quad (m, n \geq 0). \]

Now suppose that we have \(v_{4m} = 3Y^2 \ (m \geq 0)\) for some integer \(Y\). Here \(v_{4m} = 4u_m u_{2m} v_m\) since we have, by (6), \(v_{2m} = 2u_n v_n\) for all \(n\).

**Case 1.** \(m \equiv 0 \pmod{4}\). In this case \(v_m\) is a multiple of 3 by Lemma 2, and we have by Lemma 1
\[
\begin{align*}
u_m &= r^2, \\
u_{2m} &= s^2, \\
v_m &= 3t^2
\end{align*}
\]
for some non-negative integers \(r, s, t\) with \(2rst = Y\). Putting these into the relations (3) and (4) (both with \(n = m\)) gives
\[
r^4 - 18t^4 = 1 \quad \text{and} \quad s^4 = r^4 + 18t^4.
\]
Eliminating \(t\) from these equations, we thus obtain the equation
\[ s^4 = 2r^4 - 1. \]

W. Ljunggren \[\text{[2, \S 2]}\] has proved that the only solutions in positive integers (or, equivalently, non-negative integers) \(r, s\) of the equation (7) are
\[
(r, s) = (1, 1) \quad \text{and} \quad (13, 239);
\]
the former of these will give \(t = 0\), so that \(v_m = 0, m = 0, Y = 0\) and hence \(x = 0\), and the latter does not satisfy our requirement and there are no corresponding solutions \(x\).

**Case 2.** \(m \equiv 2 \pmod{4}\). By Lemma 2 \(u_m\) is then divisible by 3 and we have, by Lemma 1 again,
\[
\begin{align*}
u_m &= 3r^2, \\
u_{2m} &= s^2, \\
v_m &= t^2
\end{align*}
\]
for some positive integers \(r, s, t\) with \(2rst = Y\). We have, by (4) (with \(n = m\)), \(s^2 = 9r^4 + 2t^4\), which is obviously impossible, since g.c.d. \((t, 3) = 1\) by Lemma 1, and 2 is a (uniques) quadratic non-residue \(\pmod{3}\).

**Case 3.** \(m \equiv 1 \pmod{2}\). In this case \(u_{2m}\) is a multiple of 3 by Lemma 2, and we have, by Lemma 1,
\[
\begin{align*}
u_m &= r^2, \\
u_{2m} &= 3s^2, \\
v_m &= t^2
\end{align*}
\]
for some positive integers \(r, s, t\) with \(2rst = Y\). The relations (3) and (4) (with \(n = m\)) will yield the equations

\[ r^4 - 2t^4 = -1 \quad \text{and} \quad 3s^2 = r^4 + 2t^4, \]

whence

\[ 3s^2 - 2r^4 = 1. \] (8)

By a theorem of Ljunggren [1, Satz 3] the equation (8) has at most one solution in positive integers \(r, s\); hence, \(r = s = 1\) is the unique positive solution of (8), giving \(t = 1\), \(u_m = v_m = 1\) and so \(m = 1\). Hence we have \(v_1 = v_t = 12\), \(x = 6Y^2 = 2v_t = 24\).

Solutions with \(x \equiv 1\) (mod 3). Write \(x = 3x_1 + 1\). Then we have

\[ y^2 = (3x_1 + 1)(3x_1^2 + 2x_1 + 1), \]

where \(d = \text{g.c.d.}(3x_1 + 1, 3x_1^2 + 2x_1 + 1) = 1\) or 2.

If \(3x_1 + 1\) is odd, then \(d = 1\) and we have \(3x_1 + 1 = Y\), \(3x_1^2 + 2x_1 + 1 = X^2\) for some integers \(X, Y\) with \(\text{g.c.d.}(X, Y) = 1\), and elimination of \(x_1\) will yield the equation

\[ 3X^2 - Y^4 = 2. \] (9)

This equation has an obvious solution \(X = Y = 1\), and we find by applying a theorem of Ljunggren [3, Satz II] that \(X = Y = 1\) is the unique positive solution of (9), and this gives the solution \(x = Y^2 = 1\) of the equation (1).

If \(3x_1 + 1\) is even, then \(d = 2\) and we have \(3x_1 + 1 = 2Y^2\), \(3x_1^2 + 2x_1 + 1 = 2X^2\) for some integers \(X, Y\) with \(\text{g.c.d.}(X, Y) = 1\); but this is impossible since the congruence \(2Y^2 \equiv 1\) (mod 3) has no solutions in \(Y\).

Solutions with \(x \equiv 2\) (mod 3). Put \(x = 3x_1 - 1\). Then we have

\[ y^2 = (3x_1 - 1)(3x_1^2 - 2x_1 + 1), \]

where \(\text{g.c.d.}(3x_1 - 1, 3x_1^2 - 2x_1 + 1) = 1\) or 2.

Since \(3x_1 - 1 = Y\) is impossible in integers \(x, Y\), we must have \(3x_1 - 1\) even, and so \(3x_1 - 1 = 2Y^2\), \(3x_1^2 - 2x_1 + 1 = 2X^2\) for some integers \(X, Y\) with \(\text{g.c.d.}(X, Y) = 1\), whence

\[ 3X^2 - 2Y^4 = 1. \] (10)

The equation (10), which is satisfied by \(X = Y = 1\), has at most one solution in positive integers \(X\) and \(Y\), again by Ljunggren’s [3, Satz II]. Hence, \(X = Y = 1\) is the unique positive solution of (10), and so \(x = 2Y^2 = 2\) is the only integer solution of the equation (1) with \(x \equiv 2\) (mod 3).

The proof of our theorem is now complete.

References

[3] ---: Ein Satz über die diophantische Gleichung \(Ax^2 - By^4 = C(C = 1, 2, 4)\). Tolfte Skandinaviska Matematikerskongressen i Lund (1953), pp. 188–194.