78. *On the Limit Distributions of Decomposable Galton-Watson Processes*

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1. Introduction. Let $Z(n)=(Z_i(n))_{1\leqq i\leqq d}$ be a d-type Galton-Watson process and $M=(m_j^i)_{1\leqq i,j\leqq d}$, its mean matrix. A type j is said to be *accessible* from type i ($i\to j$) if $m_j^{i,(n)}$, the (i,j) component of M^n, is positive for some $n\geqq 0$. If $i\to j$ and $j\to i$, then i and j are said to *communicate* with each other ($i\leftrightarrow j$). Since $\leftrightarrow$ is an equivalence relation we can decompose the set of types $\{1,2,\cdots,d\}$ into the equivalence classes $C_1, C_2, \cdots, C_N$. Accessibility is a class property, i.e., $i\to j$ for some $i\in C_\alpha$ and $j\in C_\beta$ then $i'\to j'$ for all $i'\in C_\alpha$ and $j'\in C_\beta$. This is written as $\beta\precsim\alpha$ ($\beta\prec\alpha$ if $\beta\neq\alpha$) and accessibility thus induces a partial order on the classes $C_1, C_2, \cdots, C_N$. The process $Z(n)$ is said to be *indecomposable* (resp. *decomposable*) if $N=1$ (resp. $N\geqq 2$).

Let $M_\beta^\alpha=(m_j^i)_{i\in C_\alpha, j\in C_\beta}$. Then by definition each M_α^α is irreducible. We denote, by ρ_α, the maximal eigenvalue of M_α^α. The class C_α is said to be *supercritical* if $\rho_\alpha>1$, *subcritical* if $\rho_\alpha<1$. When $\rho_\alpha=1$, C_α is said to be *critical* (resp. *final*) if the generating function associated with the class is not linear (resp. linear).

Let $e^i=(0\cdots 010\cdots 0)$, where the i'th component is 1 and the others are 0, and let P_{e^i} be the measure of the process such that $P_{e^i}(Z(0)=e^i)=1$. Decomposable Galton-Watson processes with $\max\{\rho_1,\cdots,\rho_N\}=1$ have been studied by many authors. The main contributions are the following. Ogura [2] has shown that $P_{e^i}[n^{-1}(Z_j(n))_{1\leqq j\leqq d}\leqq \boldsymbol{x}\,|\,(Z_j(n))_{1\leqq j\leqq d}\neq \boldsymbol{0}]$ converges weakly. Polin [3] has studied the case $N=2$ with C_1 a critical class and C_2 a final class and has shown that $P_{e^i}[n^{-1}(Z_j(n))_{j\in C_1}\leqq \boldsymbol{x}]$ converges weakly to a gamma distribution. Foster and Ney [1] have studied the case when $1\prec 2\prec\cdots\prec N$ and each C_α is a critical class; they have shown that $P_{e^i}[((n^{-N+\alpha-1}Z_j(n))_{j\in C_\alpha})_{\alpha=1,\cdots,N}\leqq \boldsymbol{x}\,|\,(Z_j(n))_{j\in C_N}\neq\boldsymbol{0}]$ converges weakly and characterized the limit distribution. They conjectured that their limit theorems can be extended to more general processes.

The purpose of this paper is to describe the most general limit theorems for decomposable Galton-Watson processes with $\max\{\rho_1,\cdots,\rho_N\}=1$ and characterize the limit distributions. The proofs will be given elsewhere. The process we consider in this paper is as follows;

(A. 1) $Z(n)$ is decomposable,

(A. 2) for each α, M_α^α is positively regular,

(A. 3) $\max\{\rho_1, \cdots, \rho_N\}=1$,

(A. 4) for each critical class C_α, $\sum_{i,j,k\in C_\alpha}(\partial^2 F^i/\partial s^j \partial s^k)(\mathbf{1})<\infty$, where $F(s)=(F^i(s))_{1\leq i\leq d}$ is the generating function. Finally,

(A. 5) $\alpha \prec N$ for every $\alpha\neq N$.

Assumption (A. 5) is no essential restriction for our purpose.

2. Theorems. We define

$$\text{(2.1)}\quad \nu(\beta, \alpha)=\begin{cases} \max\limits_{\beta=\alpha_1\prec\alpha_2\prec\cdots\prec\alpha_k=\alpha} \#\{\alpha_i : \rho_{\alpha_i}=1\}, & \text{if } \beta\prec\alpha,\\ 1, & \text{if } \beta=\alpha \text{ and } \rho_\alpha=1,\\ 0, & \text{otherwise,}\end{cases}$$

and

(2.2) $\nu(\alpha)=\nu(\alpha, N)$.

Then we can show

Lemma 1. *If $\nu(\beta,\alpha)=0$, then*

(2.3) $(M^n)^\alpha_\beta=O(\rho^n)$ *for some* $0<\rho<1$.

If $\nu(\beta,\alpha)\geqq 1$, then there exists

(2.4) $\lim\limits_{n\to\infty} n^{-\nu(\beta,\alpha)+1}(M^n)^\alpha_\beta = M^{*\alpha}_\beta > O$.

We first state a limit theorem for the process starting from a final class. Set $D(1)=\{i\in C_\alpha\,;\,\nu(\alpha)\geqq 2\}$, $D(2)=\{i\in C_\alpha\,;\,\nu(\alpha)=1\}$ and $d_i=\#D(i)$, $i=1,2$.

Theorem 1. *Assume that C_N is a final class. Then for each $i\in C_N$ and $t>0$ there exists*

$$\text{(2.5)}\quad \lim_{n\to\infty} E_{e_i}\left[\exp\left(-\sum_{\alpha=1}^{N}\sum_{j\in C_\alpha} n^{-\nu(\alpha)+1}\lambda_j Z_j([nt])\right)\right]=G(t,\lambda)\,;$$

the limit is independent of i. G can be decomposed as follows;

(2.6) $G(t,\lambda)=G_1(t,\lambda)G_2(t,\lambda)$,

(2.7) $G_i(t,\lambda)=G_i(t,(\lambda_j)_{j\in D(i)})$, $i=1,2$.

(2.8) *$G_2(t,\lambda)$ is the Laplace transform of a probability measure on $Z^{d_2}_+$.*

Let $\{\alpha_1,\cdots,\alpha_k\}$ be the set of α's such that $\rho_\alpha=1$ and $\nu(\alpha)=2$. Then

$$\text{(2.9)}\quad G_1(t,\lambda)=\prod_{j=1}^{k} G_1^j(t,\lambda).$$

Each $G_1^j(t,\lambda)$ is the Laplace transform of an infinitely divisible distribution on $R^{d_1}_+$ and can be expressed as follows;

$$\text{(2.10)}\quad G_1^j(t,\lambda)=\exp\left\{-c_j\int_0^t \psi_j(s,\lambda)ds\right\},$$

where $c_j>0$ and ψ_j is the solution of

$$\text{(2.11)}\quad \begin{cases} \dfrac{d}{dt}\psi_j(t,\lambda)=-B_{\alpha_j}\psi_j(t,\lambda)^2+\sum\limits_{\substack{\beta\prec\alpha_j\\ \nu(\beta,\alpha_j)\geqq 2}}\sum\limits_{h\in C_\beta} a^j_h\lambda_h t^{n(\beta,\alpha_j)},\\ \psi_j(0,\lambda)=\sum\limits_{\substack{\beta\prec\alpha_j\\ \nu(\beta,\alpha_j)=1}}\sum\limits_{h\in C_\beta} b^j_h\lambda_h.\end{cases}$$

In the above, c_j, a^j_h and b^j_h are nonnegative numbers determined by $\{M^{*\alpha}_\beta\}$ and $n(\beta,\alpha_j)$ is a nonnegative integer determined by $\{\nu(\beta,\alpha)\}$. Moreover

(2.12) $$B_{\alpha_j}=\frac{1}{2}\sum_{k,l,m\in C_{\alpha_j}} v_k \frac{\partial^2 F^k}{\partial s^l \partial s^m}(\mathbf{1})u^l u^m,$$

with $\boldsymbol{u}^{\alpha_j}=(u^k)_{k\in C_{\alpha_j}}$ and $\boldsymbol{v}_{\alpha_j}=(v_k)_{k\in C_{\alpha_j}}$ being the right and left eigenvectors of $M^{\alpha_j}_{\alpha_j}$ such that $\sum_{h\in C_{\alpha_j}} u^h v_h=\sum_{h\in C_{\alpha_j}} u^h=1$.

We next state a limit theorem for the process starting from a critical class.

Theorem 2. *Assume that C_N is a critical class. Then for each $i\in C_N$ there exists*

(2.13) $$\lim_{n\to\infty} E_{e_i}\left[\exp\left(-\sum_{\alpha=1}^{N}\sum_{j\in C_\alpha} n^{-\nu(\alpha)}\lambda_j Z_j(n)\right)\middle|(Z_j(n))_{j\in C_N}\neq\mathbf{0}\right]=H(\lambda).$$

$H(\lambda)$ is represented in the form

(2.14) $$H(\lambda)=1-B_N(\psi(1,\lambda)-\eta(1,\lambda)).$$

In the above, B_N is defined by (2.12) *for the class C_N; $\psi(t,\lambda)$ is the solution of*

(2.15) $$\begin{cases}\dfrac{d\psi}{dt}(t,\lambda)=-B_N\psi(t,\lambda)^2+\sum_{\alpha:\nu(\alpha)\geqq 2}\sum_{j\in C_\alpha} a_j\lambda_j t^{n(\alpha)},\\ \psi(0,\lambda)=\sum_{\alpha:\nu(\alpha)=1}\sum_{j\in C_\alpha} b_j\lambda_j;\end{cases}$$

$\eta(t,\lambda)$ is the solution of

(2.16) $$\begin{cases}\dfrac{d\eta}{dt}(t,\lambda)=-B_N\eta(t,\lambda)^2-\dfrac{2}{t}\eta(t,\lambda)+\sum_{\alpha:\nu(\alpha)\geqq 2}\sum_{j\in C_\alpha} a_j\lambda_j t^{n(\alpha)},\\ \eta(0,\lambda)=0,\end{cases}$$

where a_j and b_j are nonnegative numbers determined by $\{M^{*\alpha}_{\beta}\}$ and $n(\alpha)=\nu(\alpha)-2$. Finally the relation of ψ and η is given by,

(2.17) $$\eta(t,\lambda)\leqq\psi(t,\lambda),$$

(2.18) $$\lim_{\substack{\lambda_j\to\infty\\ j\in C_\alpha:\nu(\alpha)=1}}\psi(t,\lambda)=\eta(t,\lambda)+\frac{1}{B_N t}.$$

Remark. It remains to investigate the limiting behavior for the process starting from a subcritical class. But the characterization of the limit distributions is complicated. We shall give the limiting behavior elsewhere.

References

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[3] A. K. Polin: Limit theorems for branching processes with immigration. Theory of Probability and its Applications, **22**, 746–754 (1978).