73. *Irreducible Characters of p-Solvable Groups*

By Tetsuro OKUYAMA*) and Masayuki WAJIMA**)

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1. Let G be a finite group and p a prime number. Let B be a p-block of G with a defect group D and b a p-block of $N_G(D)$ with $b^G=B$. It is conjectured in [1] that the number of irreducible complex characters in B of height 0 equals the number of those in b. In this note we shall show that this conjecture holds for p-solvable groups. A complete proof will be given elsewhere.

2. For a finite group G let $\mathrm{Ch}(G)$ (resp. $\mathrm{Irr}(G)$) denote the set of all characters (resp. irreducible characters) of G. If K is a normal subgroup of G and θ is an irreducible character of K, then we put $\mathrm{Irr}(G|\theta)=\{\chi\in\mathrm{Irr}(G)\,|\,(\chi_K,\theta)\neq 0\}$ and denote the set of all sums of elements in $\mathrm{Irr}(G|\theta)$ by $Ch(G|\theta)$. If B is a p-block of G, let $\mathrm{Irr}(B)$ be the set of irreducible characters of G in B.

The following theorem by Fong plays an important role in this note. We describe it using notation in a book of Isaacs [5, § 11].

Theorem (Fong [3]). *Let G be a finite group, K a normal p'-subgroup of G and $\theta\in\mathrm{Irr}(K)$. If θ is G-invariant, then there are a finite group $\hat{G}$, its cyclic central subgroup $\hat{K}$ and $\hat{\theta}\in\mathrm{Irr}(\hat{K})$ such that the following hold:*

(1) *there is an isomorphism $\tau: G/K\cong\hat{G}/\hat{K}$,*

(2) *for $K\subseteq H\subseteq G$ let $\hat{H}$ denote the inverse image in $\hat{G}$ of $\tau(H/K)$. For such subgroup H, there is a map $\sigma_H:\mathrm{Ch}(H|\theta)\to\mathrm{Ch}(\hat{H}|\hat{\theta})$ such that the following conditions hold for any $\chi,\psi\in\mathrm{Ch}(H/\theta)$:*

(a) $\sigma_H(\chi+\psi)=\sigma_H(\chi)+\sigma_H(\psi)$

(b) $(\chi,\psi)=(\sigma_H(\chi),\sigma_H(\psi))$

(c) $\sigma_G(\psi^G)=(\sigma_H(\psi))^{\hat{G}}$.

(3) *In (2), if b is a p-block of H such that $\mathrm{Irr}(b)\subseteq\mathrm{Irr}(H|\theta)$, then $\sigma_H(\mathrm{Irr}(b))=\mathrm{Irr}(\hat{b})$ for some p-block $\hat{b}$ of $\hat{H}$. Furthermore b and $\hat{b}$ have isomorphic defect groups and σ_H gives a $1-1$ height preserving correspondence between $\mathrm{Irr}(b)$ and $\mathrm{Irr}(\hat{b})$.*

The following result gives a connection between the above correspondence and Brauer's block correspondence.

Corollary. *In (3) in the above theorem assume that $DC_G(D)\subseteq H$*

*) Department of Mathematics, Faculty of Sciences, Osaka City University.
**) Department of Mathematics, Faculty of Sciences, Hokkaido University.

where D is a defect group of b. Then b^G and $\hat{b}^{\hat{G}}$ are defined and $\widehat{b^G}=\hat{b}^{\hat{G}}$.

Proof. Let D_1 be a defect group of $\hat{b}$ and put $B=b^G$. From the proof of Lemma (2C) [3], it follows that D_1 is a Sylow p-subgroup of $\widehat{DK}$. So $DC_G(D)\subseteq H$ implies $D_1C_{\hat{G}}(D_1)\subseteq \hat{H}$ as K is of p'-order. Thus $\hat{b}^{\hat{G}}$ is defined. Now it suffices to show $\hat{b}^{\hat{G}}$ is in fact $\hat{B}$. Let $\zeta\in \mathrm{Irr}(b)$ and $\zeta^G=\sum a_\chi \chi$. For an integer n we denote the p-part of n by n_p. Then by the result of Brauer ([2, (3A)]) we have $(\zeta^G(1))_p=\left(\sum_{\chi\in B} a_\chi\chi(1)\right)_p$. From the properties of σ_H in Fong's theorem, we have $\sigma_H(\zeta)^{\hat{G}}=\sum a_\chi\sigma_G(\chi)$ and $(\sigma_H(\zeta)^{\hat{G}}(1))_p=\left(\sum_{\sigma_G(\chi)\in\hat{B}} a_\chi\sigma_G(\chi)(1)\right)_p$. Thus again by the result of Brauer it follows that $\hat{b}^{\hat{G}}=\hat{B}$ and the theorem is proved.

Proposition. *Let G be a finite group with a Sylow p-subgroup P and a normal p'-subgroup K such that $G=KN_G(P)$. Let $\theta\in\mathrm{Irr}(K)$ be G-invariant. Put $N=N_G(P)$ and $L=N\cap K$. Then there is a unique $\phi\in\mathrm{Irr}(L)$ such that $(\theta_L,\phi)\not\equiv 0 \pmod p$ and the number of $\chi\in\mathrm{Irr}(G\,|\,\theta)$ such that $\chi(1)\not\equiv 0 \pmod p$ equals the number of $\psi\in \mathrm{Irr}(N\,|\,\phi)$ such that $\psi(1)\not\equiv 0\pmod p$.*

Proof. The existence and the uniqueness of ϕ follow from the result of Glauberman [4]. From Corollary 6.28 [5] there is a unique extension $\theta_0\in\mathrm{Irr}(PK)$ of θ such that p does not divide $|\det(\theta_0)|$. Also there is a unique extension $\phi_0\in\mathrm{Irr}(PL)$ such that p does not divide $|\det(\phi_0)|$. It is easily proved that if θ is extendible to G then so is θ_0 and if ϕ is extendible to N then so is ϕ_0. First we claim that $|\mathrm{Irr}(G\,|\,\theta_0)|=|\mathrm{Irr}(N\,|\,\phi_0)|$. This follows from

(*) Assume G/PK is abelian. Then θ is extendible to G if and only if ϕ is extendible to N.

We shall prove (*) by induction on the order of G. Let M be a p-complement in N. As M/L is abelian, there is a subgroup U with $L\subseteq U\subseteq M$ such that M/U is cyclic and $C_{P/P'}(U)\neq 1$. Assume $C_{P/P'}(U)=P/P'$. Then $C_P(U)=P$ and every irreducible character in $\mathrm{Irr}(UK\,|\,\theta)$ or $\mathrm{Irr}(U\,|\,\phi)$ is P-invariant. Furthermore there is a $1-1$ correspondence between $\xi\in\mathrm{Irr}(UK\,|\,\theta)$ and $\eta\in\mathrm{Irr}(U\,|\,\phi)$ such that $(\xi_U,\eta)\not\equiv 0\pmod p$ by Theorem 13.1 and 13.29 [5]. If ϕ is extendible to N, so is to M. Let $\hat{\phi}$ be an extension of ϕ to M and let $\eta=\hat{\phi}_U$. Then $|\mathrm{Irr}(UK\,|\,\theta)|=|\mathrm{Irr}(U\,|\,\phi)|=|U/L|=|UK/K|$. So θ is extendible to UK. If $\xi\in\mathrm{Irr}(UK\,|\,\theta)$ such that $(\xi_U,\eta)\not\equiv 0 \pmod p$, then ξ is MK-invariant since η is M-invariant. As MK/UK is cyclic, ξ is extendible to MK and therefore θ is extendible to MK. Then θ is extendible to G by Corollary 11.31 [5]. Conversely if θ is extendible to G, then the similar argument as above shows that ϕ is extendible to N. Next assume $C_{P/P'}(U)=Q/P'\neq P/P'$. As U is normal in M, Q and QK are normal in N and G

respectively. Let $H=QMK$ and $J=C_K(Q)$. There is a unique $\psi\in\mathrm{Irr}(J)$ such that $(\theta_J,\psi)\not\equiv 0 \pmod p$ and $(\psi_L,\phi)\not\equiv 0 \pmod p$ by the result of Glauberman [4]. Considering a group $N_G(Q)/Q$ we have by induction that ϕ is extendible to N if and only if ψ is extendible to $N_G(Q)$. Also by induction we have that ψ is extendible to $N_H(Q)$ if and only if θ is extendible to H. As $|G:H|$ and $|N_G(Q):N_H(Q)|$ are powers of p, we can conclude from Corollary 11.31 [5] that ϕ is extendible to N if and only if θ is extendible to G. Thus $(*)$ is proved.

Our claim that $|\mathrm{Irr}(G|\theta_0)|=|\mathrm{Irr}(N|\phi_0)|$ follows from $(*)$ and the result of Gallagher (see [5, Exercise 11.10]). Now we can prove the proposition. Let λ be a linear character of P. It suffices to show that $|\mathrm{Irr}(G|\theta_0\lambda)|=|\mathrm{Irr}(N|\phi_0\lambda)|$. We may assume λ is G-invariant and then the result follows from the above claim. Thus the proposition is proved.

3. In this section we shall prove our main theorem.

Theorem. *Let G be a p-solvable group. Let B be a p-block of G with a defect group D and b a p-block of $N_G(D)$ with $b^G=B$. Then the number of irreducible characters in B of height 0 equals the number of those in b.*

Proof. The result is proved by induction on the index $|G:O_{p'}(G)|$. First we consider the case that the subgroup $H=N_G(D)O_{p'}(G)$ is properly contained in G. Let $\theta\in\mathrm{Irr}(O_{p'}(G))$ such that $\mathrm{Irr}(B)\subseteq\mathrm{Irr}(G|\theta)$. If θ is not G-invariant, then the result follows from the result of Fong ([3, Theorem (2B)]) and by induction. If θ is G-invariant, then by Theorem of Fong and Corollary in § 2 we may assume that $O_{p'}(G)$ is contained in the center of G and D is a Sylow p-subgroup of G. Put $P=O_p(\mathrm{G})$. Since the kernel of every irreducible character of G and $N_G(D)$ contains P', we may assume that P is abelian. Let K be a p-complement of $O_{p,p'}(G)$. If $N_G(K)=G$, then $G=K\times P$ and the result follows easily. So we may assume $N_G(K)\neq G$. Put $Q=[P,K]$. Then $G\triangleright Q\neq 1$ and $G=N_G(K)Q$, $Q\cap N_G(K)=1$. Let $\lambda\in\mathrm{Irr}(Q)$. From the method of Wigner (see [6, Proposition 2.5]) $\mathrm{Irr}(G|\lambda)$ is obtained as follows. Put $N=N_G(K)$ and $N_1=I_N(\lambda)$, the inertia subgroup of λ in N. Let $\tilde{\lambda}$ be an extension of λ to N_1Q. Then $\mathrm{Irr}(G|\lambda)=\{(\tilde{\lambda}\zeta)^G\,|\,\zeta\in\mathrm{Irr}(N_1)\subseteq\mathrm{Irr}(N_1Q)\}$. Thus the theorem follows by applying the induction hypothesis to certain subgroups of N. Next we consider the case that $N_G(D)O_{p'}(G)=G$. Set $K=O_{p'}(G)$ and $L=K\cap N_G(D)$. There is $\theta\in\mathrm{Irr}(K)$ such that $\mathrm{Irr}(B)\subseteq\mathrm{Irr}(G|\theta)$. The result of Glauberman [4] there is a unique $\phi\in\mathrm{Irr}(L)$ such that $(\theta_L,\phi)\not\equiv 0 \pmod p$. It is clear that $\mathrm{Irr}(b)\subseteq\mathrm{Irr}(N|\phi)$. If θ is not G-invariant, then as in the above the result follows. If θ is G-invariant, then D is a Sylow p-subgroup and we have $\mathrm{Irr}(B)=\mathrm{Irr}(G|\theta)$ and $\mathrm{Irr}(b)=\mathrm{Irr}(N|\phi)$. Then the theorem follows from Proposition in § 2. Thus the theorem is proved.

4. We remark that a similar argument as in §§2–3 gives the following

Theorem. *Let π be a set of primes. If G is a finite π-solvable group and S is a Hall π-subgroup of G, then the number of irreducible characters of G of degree not divisible by any prime in π equals the number of those of $N_G(S)$.*

The above theorem for π'-solvable groups was proved by Wolf in [7]. Therefore combining our theorem with Wolf's, we see that the theorem also holds for π-separable groups.

References

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