72. Homotopy Classification of Connected Sums of Sphere Bundles over Spheres. I*)

By Hiroyasu ISHIMOTO
Department of Mathematics, Faculty of Science, Kanazawa University

(Communicated by Shokichi IYANAGA, M. J. A., Oct. 12, 1979)

1. Statement of Results. Let A be a p-sphere bundle over a q-sphere $(p, q>1)$ which admits a cross-section, and consider the following diagram which is commutative up to sign.

$$\begin{array}{ccccc} & \overset{\partial}{\nearrow} & \pi_{q-1}(SO_p) & \xrightarrow{i_*} & \pi_{q-1}(SO_{p+1}) \\ \pi_q(S^p) & & \downarrow J & & \downarrow J \\ & \underset{P}{\searrow} & \pi_{p+q-1}(S^p) & \xrightarrow{E} & \pi_{p+q}(S^{p+1}) \end{array}$$

Here, $P=[\ ,\iota_p]$ means the Whitehead product with the orientation generator ι_p of $\pi_p(S^p)$. We denote the characteristic element of A by $\alpha(A)$. Let $\alpha(A)=i_*\xi$, $\xi\in\pi_{q-1}(SO_p)$. Then, $\{J\xi\}\in J\pi_{q-1}(SO_p)/P\pi_q(S^p)$ does not depend on the choice of ξ. We denote it by $\lambda(A)$ (James-Whitehead [4]).

Let A_i, $i=1,2,\cdots,r$, be p-sphere bundles over q-spheres which admit cross-sections. It is understood that each A_i also denotes the total space of the bundle and has the differentiable structure induced from those of the fibre and the base space. $\#_{i=1}^r A_i$ means the connected sum $A_1\# A_2\#\cdots\# A_r$.

As an extension of James-Whitehead [4], we have the following

Theorem 1. *Let A_i, A_i', $i=1,2,\cdots,r$, be p-sphere bundles over q-spheres which admit cross-sections, and assume that $2p>q+1$, $q>1$, $p\neq q$. Then, the connected sums $\#_{i=1}^r A_i$, $\#_{i=1}^r A_i'$ are of the same homotopy type if and only if there exists a unimodular $(r\times r)$-matrix L of integer components such that*

$$\begin{pmatrix}\lambda(A_1')\\ \vdots \\ \lambda(A_r')\end{pmatrix}=L\begin{pmatrix}\lambda(A_1)\\ \vdots \\ \lambda(A_r)\end{pmatrix},$$

where the abelian group $J\pi_{q-1}(SO_p)/P\pi_q(S^p)$ is considered as a left Z-module.

Furthermore, we have the following

Theorem 2. *Even if $2p=q+1$, the conclusion of Theorem 1 holds also if p is odd and $p, q>1$.*

Let $p=q$. In this case, $\lambda(A_i)$, $\lambda(A_i')$ belong to $J\pi_{p-1}(SO_p)/P\pi_p(S^p)$

*) Dedicated to Professor A. Komatu for his 70th birthday.

$\cong J\pi_{p-1}(SO)\cong Z/mZ$. Here,

$$m=\begin{cases}1 & \text{if } p=3,5,6,7 \pmod 8,\\ 2 & \text{if } p=1,2 \pmod 8,\\ m(2s) & \text{if } p=0,4 \pmod 8, \text{ i.e. } p=4s\ (s>0),\end{cases}$$

where $m(2s)$ is the denominator of $B_s/4s$ (Adams [1]). Represent $\lambda(A_i)$, $\lambda(A'_i)$ by the integers λ_i, λ'_i respectively such that $0\leq\lambda_i$, $\lambda'_i\leq m-1$.

Theorem 3. *Let A_i, A'_i, $i=1,2,\cdots,r$ $(r>1)$, be p-sphere bundles over p-spheres $(p>2, p\neq 4, 8)$. Then, the connected sums $\#_{i=1}^r A_i$, $\#_{i=1}^r A'_i$ are of the same homotopy type if and only if G.C.D. $(\lambda_1,\cdots,\lambda_r, m)$ $=$ G.C.D. $(\lambda'_1,\cdots,\lambda'_r, m)$. Especially, if $p=1,2 \pmod 8$, then $m=2$ and therefore, the connected sums are of the same homotopy type if and only if they have simultaneously non-trivial bundles or only trivial bundles.*

The following is equivalent to Theorem 3 for $r>1$, but remains valid also for $r=1$. This gives an analogue of Theorem 1 in the case $p=q$ $(p>2)$.

Theorem 3′. *Let A_i, A'_i, $i=1,2,\cdots,r$, be p-sphere bundles over p-spheres $(p>2, p\neq 4,8)$. Then, the connected sums $\#_{i=1}^r A_i$, $\#_{i=1}^r A'_i$ are of the same homotopy type if and only if there exists a unimodular $(r\times r)$-matrix L of integer components such that*

$$\begin{pmatrix}\lambda(A'_1)\\ \vdots\\ \lambda(A'_r)\end{pmatrix}=L\begin{pmatrix}\lambda(A_1)\\ \vdots\\ \lambda(A_r)\end{pmatrix}.$$

Remark. If $p=3,5,6,7 \pmod 8$, the theorems hold trivially, since $\pi_{p-1}(SO_{p+1})=0$.

Remark. Even if $p=4,8$, the theorems hold if all $\alpha(A_i)$, $\alpha(A'_i)$ are even, that is, $\#_{i=1}^r A_i$, $\#_{i=1}^r A'_i$ are of type II (Milnor [5]).

2. Sketch of the proofs. The detailed proofs of the above theorems will appear elsewhere. We give here only an outline.

$\#_{i=1}^r A_i$ has the cell structure $\{\bigvee_{i=1}^r (S_i^p\vee S_i^q)\}\cup_\varphi D^{p+q}$ and φ is given by $\{\varphi\}=\sum_{i=1}^r(\iota_p^i\circ\eta_i+[\iota_q^i,\iota_p^i])$, where ι_p^i, ι_q^i are the orientation generators of $\pi_p(S_i^p)$, $\pi_q(S_i^q)$ respectively, $\eta_i=J\xi_i$, and $i_*\xi_i=\alpha(A_i)$. S_i^p, S_i^q correspond respectively to the fibre of A_i and the cross-section determined by ξ_i. Let $\bar{A}_i$ be the associated $(p+1)$-disk bundle of A_i. Then, the boundary connected sum $W=\natural_{i=1}^r\bar{A}_i$ is considered as a handlebody and ∂W $=\#_{i=1}^r A_i$. Using handlebody theory of Wall [7], we see that for any basis of $H_q(\partial W)\cong H_q(W)$ $(p\neq q)$, there is a representation $\partial W=\#_{i=1}^r\tilde{A}_i$ by p-sphere bundles over q-spheres admitting cross-sections. Thus, for a given homotopy equivalence $f:\partial W=\#_{i=1}^r A_i\to\#_{i=1}^r A'_i$, we have a representation $\partial W=\#_{i=1}^r\tilde{A}_i$ such that $f_*\tilde{\iota}_p^i=\iota_p'^i$ and $j'_*(f_*\tilde{\iota}_q^i)=j'_*\iota_q'^i$ $(p<q)$, where $j'_*:\pi_q(\bigvee_{i=1}^r(S_i'^p\vee S_i'^q))\to\pi_q(\bigvee_{i=1}^r(S_i'^p\vee S_i'^q),\ \bigvee_{i=1}^r S_i'^p)$ and j' is the inclusion map. Then, comparing the attaching maps of the $(p+q)$-cells

using Hilton [3] and Barcus-Barratt [2], we see that $\lambda(\tilde{A}_i)=\lambda(A_i')$, $i=1,2,\cdots,r$. Since we have transformed the basis of $H_q(\partial W) \cong H_q(W)$, $(\lambda(\tilde{A}_i))=L(\lambda(A_i))$ for some unimodular matrix L. The converse is similar. Hence, we have Theorem 1. Theorem 2 is obtained by comparing more precisely the attaching maps of the $(p+q)$-cells.

Let $p=q$ $(p>2)$, and let $\nu: H=H_p(\#_{i=1}^r A_i)\to\pi_{p-1}(SO_p)$ be the map assigning to each $x\in H$ the characteristic element of the normal bundle of the imbedded p-sphere which represents x. Let $\mu=J\circ\nu$. Similarly, H', μ' are defined for $\#_{i=1}^r A_i'$. H, H' have symplectic bases and after certain calculations we see that $(\lambda(A_i))=L(\lambda(A_i'))$ for some unimodular matrix L if and only if μ, μ' have the same values on some symplectic bases of H, H'. Then, by Lemma 8 of Wall [6], we have Theorem 3.

References

[1] Adams, J. F.: On the groups $J(X)$-IV. Topology, **5**, 21–71 (1966).

[2] Barcus, W. D., and M. G. Barratt: On the homotopy classification of the extensions of a fixed map. Trans. Amer. Math. Soc., **88**, 57–74 (1958).

[3] Hilton, P. J.: On the homotopy groups of the union of spheres. J. London Math. Soc., **30**, 154–172 (1955).

[4] James, I. M., and J. H. C. Whitehead: The homotopy theory of sphere bundles over spheres. I. Proc. London Math. Soc., **4**(3), 196–218 (1954).

[5] Milnor, J. W.: On simply connected 4-manifolds. Symp. Intern. de Topologia Algebrica, Mexico, pp. 122–128 (1958).

[6] Wall, C. T. C.: Classification of $(n$-$1)$-connected $2n$-manifolds. Ann. of Math., **75**, 163–189 (1962).

[7] ——: Classification problems in differential topology-I, Classification of handlebodies. Topology, **2**, 253–261 (1963).