70. *On an Explicit Construction of Siegel Modular Forms of Genus 2*

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1. In this note, we shall present an explicit correspondence from a pair of elliptic modular forms to a Siegel modular form of genus 2, which "preserves" Euler products, by means of theta series.

We denote by $\boldsymbol{H}$ the Hamilton quaternion algebra. For a ring A, let $A^\times$ denote the group of invertible elements of A. For a square matrix M, $\sigma(M)$ denotes the trace of M. For modular forms and Euler products associated with them, we shall use notation as is given in A. N. Andrianov [1] and G. Shimura [4].

2. Let D be a definite quaternion algebra over $\boldsymbol{Q}$ whose discriminant is d^2 and R be a maximal order of D. Let $D_A^\times$ denote the adelization of $D^\times$. For a prime l, we put $D_l=D\otimes_{\boldsymbol{Q}}\boldsymbol{Q}_l$ and $R_l=R\otimes_{\boldsymbol{Z}}\boldsymbol{Z}_l$ and let ι_l denote the canonical injection of $D_l^\times$ into $D_A^\times$. Set $K=\prod_l R_l^\times \times \boldsymbol{H}^\times$ and let $D_A^\times=\bigcup_{i=1}^{H} D^\times y_i K$ be a double coset decomposition of $D_A^\times$ such that the reduced norm of y_i $(1\leq i\leq H)$ is $1\in \boldsymbol{Q}_A^\times$. For $1\leq i, j\leq H$, define a lattice L_{ij} of D by $L_{ij}=D\cap y_i\left(\prod_l R_l\right)y_j^{-1}$ and put $R_i=L_{ii}$, $e_i=|R_i^\times|$. Let N, Tr and $*$ stand for the reduced norm, the reduced trace and the main involution of D respectively. Let H_n be the Siegel upper half space of genus n. Set

$$(1)\qquad \vartheta_{ij}(z)=\sum_{x\in L_{ij}}\exp(2\pi\sqrt{-1}N(x)z),\qquad z\in H_1,$$

$$(2)\qquad \tilde{\vartheta}_{ij}(z)=\sum_{(x,y)\in L_{ij}\oplus L_{ij}}\exp\left(2\pi\sqrt{-1}\sigma\left(\begin{pmatrix}N(x) & Tr(xy^*)/2\\ Tr(xy^*)/2 & N(y)\end{pmatrix}z\right)\right),\qquad z\in H_2.$$

Then ϑ_{ij} and $\tilde{\vartheta}_{ij}$ are Siegel modular forms of genera 1 and 2 respectively. The weight of them is 2 and the level of them is d. Let $S(R)$ denote the space of complex valued functions φ on $D_A^\times$ which satisfy that $\varphi(\gamma gk)=\varphi(g)$ for any $\gamma\in D^\times$, $k\in K$, $g\in D_A^\times$. For a prime $l\nmid d$, fixing a splitting $D_l\cong M_2(\boldsymbol{Q}_l)$ such that R_l is mapped onto $M_2(\boldsymbol{Z}_l)$, we put

$$(T'(l)\varphi)(g)=\sum_{v=0}^{l-1}\varphi\left(g\cdot\iota_l\begin{pmatrix}l & v\\ 0 & 1\end{pmatrix}\right)+\varphi\left(g\cdot\iota_l\begin{pmatrix}1 & 0\\ 0 & l\end{pmatrix}\right).$$

For $\varphi,\varphi_1,\varphi_2\in S(R)$ and $1\leq i\leq H$, set

$$(3)\qquad f_i(\varphi)=\sum_{j=1}^{H}(\varphi(y_j)/e_j)\vartheta_{ij},$$

$$(4) \qquad F(\varphi_1, \varphi_2)=\sum_{i=1}^{H}\sum_{j=1}^{H}(\varphi_1(y_i)\varphi_2(y_j)/e_ie_j)\tilde{\vartheta}_{ij}.$$

Theorem 1. *Let $\varphi \in S(R)$ and l be a prime such that $l \nmid d$. If $T'(l)\varphi=\lambda\varphi$, we have $T(l)f_i(\varphi)=\lambda f_i(\varphi)$ for any i, $1\leq i\leq H$.*

Theorem 2. *Let $\varphi_1, \varphi_2 \in S(R)$ and l be an odd prime such that $l\nmid d$. If $T'(l)\varphi_i=\lambda_i\varphi_i$, $1\leq i\leq 2$, then we have*

$$T(l)F(\varphi_1, \varphi_2)=(\lambda_1+\lambda_2)F(\varphi_1, \varphi_2),$$
$$T(l^2)F(\varphi_1, \varphi_2)=(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2-2l-1)F(\varphi_1, \varphi_2).$$

A detailed proof of Theorem 2 is given in [5].

Remark 1. Suppose that φ, φ_1 and φ_2 are common eigenfunctions of $T'(l)$ for every $l\nmid d$. Then $f_i(\varphi)$ is a cusp form if φ is not a constant function. If d is a prime, $F(\varphi_1, \varphi_2)$ is a cusp form if φ_2 is not a constant multiple of φ_1.

3. Hereafter, we shall assume that d is equal to a prime p and discuss in what condition F does not vanish. Every identity between Euler products shall mean the equality up to the 2 and p-factors. Let ϖ be a prime element of R_p and put $S^+(R)=\{\varphi \in S(R)\,|\,\varphi(g\iota_p(\varpi))=\varphi(g), {}^\forall g \in D_A^\times\}$, $S^-(R)=\{\varphi\in S(R)\,|\,\varphi(g\iota_p(\varpi))=-\varphi(g), {}^\forall g\in D_A^\times\}$. We can show easily that if $\varphi\in S^\pm(R)$, then $f_i(\varphi)\in G_2^\mp(\Gamma_0(p))$. Let $\varphi_1, \cdots, \varphi_T$ (resp. $\varphi_{T+1}, \cdots, \varphi_H$) be a basis of $S^+(R)$ (resp. $S^-(R)$) consisting of common eigenfunctions of $T'(l)$, $l\neq p$. Here T is the type number of D. Note that the integral quadratic form $x\to N(x)$ on L_{ij} represents 1 if and only if $i=j$. Hence we have $F(\varphi_i, \varphi_i)\neq 0$, which is an Eisenstein series whose Euler product is $L(s, \varphi_i)^2$. If $\varphi_i\in S^+(R)$ and $\varphi_j \in S^-(R)$, we can show $F(\varphi_i, \varphi_j)=0$. However if $i, j\leq T$ or $i, j>T$, we can expect the nonvanishing of $F(\varphi_i, \varphi_j)$. At present, we can prove the followings.

Let K be an imaginary quadratic field of class number 1 such that $\left(\frac{K}{p}\right)=-1$. Then the maximal order $\mathfrak{O}$ of K is embedded in a maximal order R_i of D.

Theorem 3. *Let the notation be as above. If $\varphi_t(y_i)\neq 0$, $\varphi_u(y_i)\neq 0$, we have $F(\varphi_t, \varphi_u)\neq 0$.*

Let K be an imaginary quadratic field of class number 2 such that $\left(\frac{K}{p}\right)=-1$. The maximal order $\mathfrak{O}$ of K is embedded in some R_i. Let $j(\mathfrak{A}_1)$, $j(\mathfrak{A}_2)$ be the singular invariants for two representatives $\mathfrak{A}_1$, $\mathfrak{A}_2$ of ideal classes of K. Let F be the real quadratic field generated over $\boldsymbol{Q}$ by $j(\mathfrak{A}_1)$.

Theorem 4. *Let the notation be as above. We assume that $\left(\frac{F}{p}\right)=-1$ and that p does not divide $j(\mathfrak{A}_1)-j(\mathfrak{A}_2)$. For φ_t, φ_u such that $t, u\leq T$* or *$t, u>T$, we have $F(\varphi_t, \varphi_u)\neq 0$ if $\varphi_t(y_i)\neq 0$ and $\varphi_u(y_i)\neq 0$.*

Proof. We shall prove Theorem 4. The proof of Theorem 3 is simpler. Since $\left(\frac{F}{p}\right)=-1$ and p does not divide $j(\mathfrak{A}_1)-j(\mathfrak{A}_2)$, by virtue of Deuring's results on supersingular moduli and a theorem of Chevalley-Hasse-Noether, $\mathfrak{O}$ is embeddable in exactly two maximal orders R_i and R_j which have non-principal two sided ideals. Moreover R_i and R_j are conjugate. We take $s, v \in \boldsymbol{Z}$ so that the roots of $X^2-sX+v=0$ generate $\mathfrak{O}$ over $\boldsymbol{Z}$. Let $\sum a(N)\exp(2\pi\sqrt{-1}\sigma(Nz))$ be the Fourier expansion of $F(\varphi_t, \varphi_u)$. If $t, u \leq T$ or $t, u > T$, we see easily that $a\left(\begin{pmatrix} 1 & s/2 \\ s/2 & v \end{pmatrix}\right)=\varphi_t(y_i)\varphi_u(y_i)\cdot$(some positive integer). Therefore $F(\varphi_t, \varphi_u)\neq 0$ if $\varphi_t(y_i)\neq 0$ and $\varphi_u(y_i)\neq 0$.

Remark 2. For $\tau \in \mathrm{Aut}(\boldsymbol{C})$ and $\varphi \in S^{\pm}(R)$, define $\varphi^{\tau} \in S^{\pm}(R)$ by $\varphi^{\tau}(g)=(\varphi(g))^{\tau}$, $g \in D_A^{\times}$. Then it is clear that $f_i(\varphi^{\tau})=(f_i(\varphi))^{\tau}$. Using this fact and finding several suitable imaginary quadratic fields of class number 1 or 2 which satisfy the conditions of Theorem 3 or 4, we obtain the followings. For every $f \in G_2^-(\Gamma_0(p))$ (resp. $f \in S_2^+(\Gamma_0(p))$ and $\tau \in \mathrm{Aut}(\boldsymbol{C})$, where f is a common eigenfunction of Hecke operators, there exist non-zero Siegel modular forms F_1 and F_2 (resp. F) such that $L(s, F_1)=L(s,f)L(s,f^{\tau})$, $L(s,F_2)=\zeta(s)\zeta(s-1)L(s,f)$ (resp. $L(s,F)=L(s,f)L(s,f^{\tau})$), if $p \leq 103$. Here ζ denotes the Riemann zeta function. If $p=31$ for example, we can obtain a Siegel modular form whose Euler product is equal to the one dimensional part of the Hasse-Weil zeta function of certain simple 2-dimensional abelian variety (cf. [4, Theorem 7.15]).

Remark 3. Put $I_i=|\{t \mid \varphi_t(y_i)\neq 0\}|$. We can show $\dim_{\boldsymbol{C}}\langle \vartheta_{ij} \mid 1 \leq j \leq H\rangle = I_i$ for every $1 \leq i \leq H$. Hence $I_i \geq 2$ if $H \geq 2$ for every i. A. Pizer [3] found for many p that $I_i=H$ if R_i has a non-principal two sided ideals and that $I_i=T$ if R_i does not. The importance of this conjecture is now obvious.

4. For the higher weight case, we can also construct (by a similar formula to (4)) a Siegel modular form F (of genus 2) of weight k from a pair (f_1, f_2) of elliptic modular forms such that the Euler product $L(s, F)$ is equal to $L(s-k+2, f_1)L(s, f_2)$, where $f_1 \in G_2(\Gamma_0(p))$, $f_2 \in S_{2k-2}^{\mathrm{new}}(\Gamma_0(p))$ and k is even. In our construction, some explicit choices of spherical functions of nice behavior are crucial. Here we content ourselves by giving an example for the case $p=3$, $k=4$. Set

$$S=\begin{pmatrix} 1 & 0 & 3/2 & 0 \\ 0 & 1 & 0 & 3/2 \\ 3/2 & 0 & 3 & 0 \\ 0 & 3/2 & 0 & 3 \end{pmatrix}$$

and

$$\Theta_1(z)=\sum_{(x,y)\in Z^4\oplus Z^4} P_1(x,y)^2 \exp(2\pi\sqrt{-1}\sigma(Q(x,y)z)),$$
$$\Theta_2(z)=\sum_{(x,y)\in Z^4\oplus Z^4} P_2(x,y)^2 \exp(2\pi\sqrt{-1}\sigma(Q(x,y)z)), \qquad z\in H_2.$$

Here we put $x=(x_1, x_2, x_3, x_4)$, $y=(y_1, y_2, y_3, y_4)$, $P_1(x, y)=(x_1y_2-y_1x_2+x_1y_4-y_1x_4-x_2y_3+y_2x_3)$, $P_2(x,y)=(x_1y_3-y_1x_3+x_2y_4-y_2x_4)$ and $Q(x,y)=\begin{pmatrix} {}^txSx & {}^txSy \\ {}^txSy & {}^tySy \end{pmatrix}$. Then we can prove, for the cusp form $F=\Theta_1-\Theta_2$, that $L(s,F)=\zeta(s-2)\zeta(s-3)L(s,f)$ holds, where f is the primitive cusp form of $S_6(\Gamma_0(3))$. Thus our results seem to "explain" some conjectural examples given in N. Kurokawa [2]. Finally we note that it is also possible to obtain similar results for Hilbert modular forms over real quadratic fields of weight $(2, 2k-2)$.

References

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