68. On the Pseudo-Parabolic Regularization of the Generalized Kortweg-de Vries Equation

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(Communicated by Kôsaku YOSIDA, M. J. A., Oct. 12, 1979)

1. Introduction. This note is concerned with the initial-boundary value problem:

(1) $u_t+(\phi(u))_x+u_{xxx}-\varepsilon u_{txx}=0,\qquad t\in R,\ x\in(0,1),$

(2) $u(0,x)=g(x),\qquad x\in(0,1),$

(3) $u(t,0)=u(t,1),\qquad t\in R,$

where $\varepsilon>0$, ϕ is a function of class $C^\infty(R)$ satisfying $\phi(0)=0$ and g is a given initial function satisfying $g(0)=g(1)$.

The pseudo-parabolic equation (1) is understood to be a generalization of model equations for long water waves of small amplitude (see for instance [1]). The equation (1) is also regarded as a regularization of the generalized Kortweg-de Vries equation

(4) $$u_t+(\phi(u))_x+u_{xxx}=0.$$

For the parabolic regularizations of the generalized KdV equation, see [4].

Here we treat the initial-boundary value problem (1)–(3) from the viewpoint of the semigroup theory and describe the properties of solutions of the problem in terms of nonlinear group in a Hilbert space.

2. Theorem. We denote by $\|\cdot\|$ the norm of the Lebesgue space $L^2(0,1)$. For each positive integer m, we write V^m for the closed subspace of the Sobolev space $H^m(0,1)$ defined by

$$V^m=\{v\in H^m(0,1)\,;\,v^{(l)}(0)=v^{(l)}(1),\ 0\leqq l\leqq m-1\}.$$

We also denote by D the differential operator d/dx from $H^1(0,1)$ into $L^2(0,1)$, i.e., D is defined by $Dv=v'$ for $v\in H^1(0,1)$.

Now we define a linear operator L_ε from V^2 into V^1 by

$$L_\varepsilon v=\frac{1}{\varepsilon}Dv\qquad\text{for } v\in V^2,$$

and a nonlinear operator F_ε on V^1 by

$$[F_\varepsilon v](x)=\int_0^1 K_\varepsilon(x,\xi)\left\{\phi(v(\xi))+\frac{1}{\varepsilon}v(\xi)\right\}d\xi$$

for $v\in V^1$ and $x\in[0,1]$, where

$$K_\varepsilon(x,\xi)=\frac{\operatorname{sgn}(x-\xi)}{2(1-e)\varepsilon}\left\{\exp\left(\frac{|x-\xi|}{\sqrt{\varepsilon}}\right)-\exp\left(1-\frac{|x-\xi|}{\sqrt{\varepsilon}}\right)\right\}\quad\text{for } x,\xi\in[0,1].$$

Note that $w\equiv F_\varepsilon v$ gives a unique solution of the boundary value problem

$$\varepsilon w'' - w = (\phi(v))' + \frac{1}{\varepsilon} v' ; \ w(0) = w(1), \ w'(0) = w'(1).$$

We then see that L_ε is the infinitesimal generator of a linear group $\{U_\varepsilon(t) ; t \in R\}$ of isometries on the Hilbert space V^1. Also, we see that F_ε is Fréchet differentiable over V^1 and Lipschitz continuous on each bounded subset of V^1 (cf. [3]). In view of these facts, we see that $L_\varepsilon + F_\varepsilon$ generates a nonlinear group $\{G_\varepsilon(t) ; t \in R\}$ of C^1-diffeomorphisms on V^1 (cf. [2]). More precisely, we have the following result.

Theorem. *For each $\varepsilon > 0$, there exists a nonlinear group $\{G_\varepsilon(t) ; t \in R\}$ of C^1-diffeomorphisms on V^1 satisfying the following properties:*

(i) $G_\varepsilon(t)g = U_\varepsilon(t)g + \int_0^t U_\varepsilon(t-s)F_\varepsilon(G_\varepsilon(s)g)ds$ *for* $t \in R$ *and* $g \in V^1$.

(ii) *If* $g \in V^2$, *then* $G_\varepsilon(t)g$ *is of class* $C^1(R ; V^1)$ *as a* V^1-*valued function on* R *and*

$(d/dt)G_\varepsilon(t)g = (L_\varepsilon + F_\varepsilon)G_\varepsilon(t)g = dG_\varepsilon(t ; g)(L_\varepsilon + F_\varepsilon)g$ *for* $t \in R$,

where $dG_\varepsilon(t ; g)$ *denotes the Fréchet derivative of* $G_\varepsilon(t)$ *at* g.

(iii) *Each of* $G_\varepsilon(t)$ *maps* V^m *into itself for* $m \geqq 1$.

(iv) *Let* $g \in V^2$ *and set* $u(t, x) = [G_\varepsilon(t)g](x)$ *for* $(t, x) \in R \times [0, 1]$. *Then* u *gives a solution of the problem* (1)–(3) *in the sense that* u *satisfies the equality*

$$\int_0^1 \{u_t(t, x)w(x) + (\phi(u(t, x)))_x w(x) - u_{xx}(t, x)w'(x) + \varepsilon u_{tx}(t, x)w'(x)\}dx = 0$$

for every $t \in R$ *and* $w \in V^1$. *If in particular,* $g \in V^4$, *then* u *satisfies the equation* (1) *pointwise on* $R \times (0, 1)$.

(v) $\|G_\varepsilon(t)g\|^2 + \varepsilon \|DG_\varepsilon(t)g\|^2 = \|g\|^2 + \varepsilon \|Dg\|^2$ *for* $t \in R$ *and* $g \in V^1$.

(vi) $\|DG_\varepsilon(t)g\|^2 - 2\int_0^1 \psi([G_\varepsilon(t)g](x))dx = \|Dg\|^2 - 2\int_0^1 \psi(g(x))dx$

for $t \in R$ *and* $g \in V^1$, *where* $\psi(\xi) = \int_0^\xi \phi(\tau)d\tau$ *for* $\xi \in R$.

(vii) *For each* $m \geqq 2$, *there is a monotone increasing function* $\alpha_m : [0, \infty) \mapsto [0, \infty)$ *such that*

$$\|G_\varepsilon(t)g\|^2 + \|D^m G_\varepsilon(t)g\|^2 + \varepsilon \|D^{m+1}G_\varepsilon(t)g\|^2 \leqq \alpha_m(\|g\|^2 + \varepsilon \|Dg\|^2 + \|D^m g\|^2 + \varepsilon \|D^{m+1}g\|^2 + |t|)$$

for $t \in R$ *and* $g \in V^{m+1}$.

We refer to [2] for the proof of the assertions (i) and (ii). The properties (iii)–(vi) are obtained in a manner similar to [3]; and the estimates in (vii) are established by solving Bellman-Bihari integral inequalities.

In view of the properties (v) and (vii), it can be shown that if $g \in V^4$ then $G_\varepsilon(t)g$ converges as $\varepsilon \to 0+$ to a function in the space $C([-T, T] ; L^2(0, 1))$, for every $T > 0$; and the limit function furnishes a "solution" of the generalized KdV equation (4). For the detailed argument concerning above facts, we shall publish it elsewhere.

References

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