66. *On the Existence of Solutions for Linearized Euler's Equation*

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1. **Statement of results.** Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$ and ν be the unit exterior normal to $\partial\Omega$. We denote by H the real Hilbert space consisting of all the real vector fields u with coefficients in $L^2(\Omega)$ such that $\operatorname{div} u=0$ in Ω and $u\cdot\nu=0$ on $\partial\Omega$, and set $V=H\cap(H^1(\Omega))^n$. Denoting by P the orthogonal projection from $(L^2(\Omega))^n$ onto H, we consider the following initial value problem:

$$\text{(I.V.P.)}\qquad \begin{cases} \dfrac{du}{dt}+P(a,\text{grad})u=f, \\ \qquad\qquad u(0)=u_0, \end{cases}$$

where $f=f(t)$ and $a=a(t)$ are given H-valued functions and u_0 is an element in H. (a, grad) denotes $\sum_{j=1}^{n} a^j(x,t)\partial/\partial x_j$. Our aim in this note is to establish the existence and uniqueness of the solution for (I.V.P.) under certain mild assumptions on data. As a byproduct, we have proved the essential self-adjointness of $iP(a, \text{grad})$ as an operator on H when a does not depend on t. When $a=u$, (I.V.P.) is the initial value problem for Euler's equation of incompressible ideal fluids. However, we could not take a and u from the same function space (see Theorem 2 below). We note that nothing is known about the existence of global weak solutions for Euler's equation when $n\geqslant 3$.

Our method of proof is based on the "vanishing viscosity" argument for the following problem:

$$\text{(I.V.P.)}_\varepsilon\qquad \begin{cases} \dfrac{du}{dt}+\varepsilon Nu+P(a,\text{grad})u=f, \\ \qquad\qquad u(0)=u_0, \end{cases}$$

where N denotes the Laplacian, $-\Delta$, acting on 1-forms with the Neumann boundary condition: $u\cdot\nu=0$, $(du)_{\text{norm}}=0$ on $\partial\Omega$ which is associated with the bilinear form: $(du,dv)+(\delta u,\delta v)$, defined on $\{u\in(H^1(\Omega))^n;\ u\cdot\nu=0, \text{ on } \partial\Omega\}$, and $\varepsilon>0$ is a constant. Here we have denoted by d the exterior differentiation and by δ its formal adjoint. (Throughout this paper, vector fields and 1-forms are identified by means of Euclidean metric.) See [4] or [5] for the details of the Neumann problem for differential forms. It is easy to see that N

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defines a non-negative self-adjoint operator on H. See also [2]. Now our results are as follows.

Theorem 1. *Let T be a fixed positive number. Then, for each $\varepsilon>0$, $u_0 \in V$, $f \in L^2(0,T;V)$ and $a \in L^\infty(0,T;H \cap (W^{1,\infty}(\Omega))^n)$,* (I.V.P.)$_\varepsilon$ *admits a unique solution u_ε such that*

(i) $u_\varepsilon \in C([0,T];V) \cap L^2(0,T;(H^2(\Omega))^n)$,

(ii) $du_\varepsilon/dt \in L^2(0,T;H)$.

Theorem 2. *Under the assumptions of Theorem* 1, *each subsequence of $\{u_\varepsilon\}$ converges weak-star in $L^\infty(0,T;V)$ and strongly in $L^2(0,T;H)$ to a unique solution u of* (I.V.P.) *such that*

(a) $\|u(t)\|_V^2 \leqslant \exp\{C(a)t\}(\|u_0\|_V^2 + \int_0^T \exp\{-C(a)s\}\|f(s)\|_V^2 ds)$ *for a.e.* $t \in [0,T]$,

(b) $du/dt \in L^2(0,T;H)$,

(c) $\|u(t)\|_H^2 = \|u_0\|_H^2 + 2\int_0^T (f(s),u(s))ds$, *for each* $t \in [0,T]$,

where $C(a)>0$ is a constant depending on the norm of a as an element in $L^\infty(0,T;(W^{1,\infty}(\Omega))^n)$.

It should be noticed that the above theorems remain valid if we replace $P(a,\mathrm{grad})$ by $-P(a,\mathrm{grad})$, which amounts to solving (I.V.P.) backwards in time. Thus, when a is independent of t, estimates (a) and (b) with $f=0$ together with Stone's theorem imply the essential self-adjointness of $iP(a,\mathrm{grad})$ restricted to V. More specifically we have obtained

Theorem 3. *For each $a \in H \cap (W^{1,\infty}(\Omega))^n$, the linear operator $iP(a,\mathrm{grad})$ is essentially self-adjoint on $D(N)$.*

It seems to us that the above result has some relations with a conjecture of E. Nelson concerning the self-adjointness of the Liouville operator (see [1] and [6]).

As is mentioned before, we do not know whether, in Theorem 2, it is possible to take a and u from $L^\infty(0,T;V)$ or not, even when $n=3,4$. In this connection we have obtained

Theorem 4. *For each $u_0 \in H$, $a \in L^\infty(0,T;H)$ and $f \in L^2(0,T;H)$, there exists at least one function u in $L^\infty(0,T;H)$ satisfying* (I.V.P.) *in the following sense: The identity*

$$-\int_0^T (u(t),v)h'(t)dt - \int_0^T (u(t),(a(t),\mathrm{grad})v)h(t)dt$$
$$=(u_0,v)h(0) + \int_0^T (f(t),v)h(t)dt,$$

is valid for each $v \in (C_0^\infty(\Omega))^n$ with $\operatorname{div} v=0$, *and each $h \in C^1([0,T];R)$ with $h(T)=0$.*

Finally, it is to be noticed that our results except Theorem 4 can not be obtained if we use, instead of N, the Stokes operator, $-P\Delta$, with the Dirichlet boundary condition.

2. Sketch of proof. First we prove Theorem 1 by the method of Faedo-Galerkin. Let $\{w_j\}$ be a total set of linearly independent vectors in V, and choose $u_{0,m}=\sum_{j=1}^{m} g_{jm}^0 w_j$ such that $u_{0,m}\to u_0$ in V and $\|u_{0,m}\|_V \leqslant \|u_0\|_V$. We determine $u_m(t)=\sum_{j=1}^{m} g_{jm}(t)w_j$ by the equations:

$$(1)\quad \begin{aligned} &(u'_m, w_j)+\varepsilon(du_m, dw_j)+((a, \mathrm{grad})u_m, w_j)=(f, w_j), \qquad 1\leqslant j\leqslant m,\\ &u_m(0)=u_{0,m}. \end{aligned}$$

If we multiply (1) by $g_{jm}(t)$ and add these equations for j, then, since $((a, \mathrm{grad})u_m, u_m)=0$, we get

$$\begin{aligned} (d/dt)\,\|u_m(t)\|_H^2+2\varepsilon\,\|du_m(t)\|_{L^2}^2 &= 2(f(t), u_m(t))\\ &\leqslant (1/\varepsilon)\,\|f(t)\|_{V'}^2+\varepsilon\,\|u_m(t)\|_V^2\\ &=(1/\varepsilon)\,\|f(t)\|_{V'}^2+\varepsilon(\|du_m(t)\|_{L^2}^2+\|u_m(t)\|_H^2). \end{aligned}$$

Integrating this for t we easily derive the boundedness of $\{u_m\}$ in $L^\infty(0, T; H)\cap L^2(0, T; V)$. Note that here we have used the coerciveness of the Neumann problem (see [5]). Similarly, by multiplying (1) by $g'_{jm}(t)$ and adding for j, we can deduce the boundedness of $\{u'_m\}$ in $L^2(0, T; H)$. Thus we may choose a subsequence of $\{u_m\}$ converging weak-star in $L^\infty(0, T; H)$ and weakly in $L^2(0, T; V)$ to an element u_ε such that $u'_\varepsilon \in L^2(0, T; H)$ and

$$(2)\quad (u'_\varepsilon, v)+\varepsilon(du_\varepsilon, dv)+((a, \mathrm{grad})u_\varepsilon, v)=(f, v), \text{ a.e. in } (0, T),$$

for each $v\in V$. Since $v\to(f-(a, \mathrm{grad})u_\varepsilon-u'_\varepsilon, v)$ is continuous in L^2 topology and V is dense in H (see [2]), it follows from the coerciveness of the Neumann problem that $u_\varepsilon \in L^2(0, T; (H^2(\Omega))^n)$, so that $u_\varepsilon\in C([0, T]; V)$ by an interpolation theorem. The proof of the uniqueness is standard, so omitted.

Next we prove Theorem 2. By virtue of the well-known orthogonal decomposition theorem for $(L^2(\Omega))^n$ (see [7, Chap. I]), there exists a distribution $p_\varepsilon(x, t)$ on $\Omega\times(0, T)$ such that

$$(3)\quad \partial u_\varepsilon/\partial t-\varepsilon\Delta u_\varepsilon+(a, \mathrm{grad})u_\varepsilon+\mathrm{grad}\, p_\varepsilon=f, \text{ in } \Omega\times(0, T).$$

Applying exterior differentiation to (3) we have

$$(4)\quad \partial(du_\varepsilon)/\partial t-\varepsilon\Delta(du_\varepsilon)+(a, \mathrm{grad})du_\varepsilon=df+R(a, D)u_\varepsilon,$$

where $R(a, D)$ is a homogeneous first order differential operator whose coefficients are linear combinations of $\partial a^j/\partial x_k$. From this and the fact that du_ε satisfies the Neumann condition for 2-forms we obtain

$$(5)\quad (d/dt)\,\|du_\varepsilon(t)\|_{L^2}^2\leqslant C(a)\,\|u_\varepsilon(t)\|_V^2+\|df(t)\|_{L^2}^2.$$

Since $(d/dt)\,\|u_\varepsilon(t)\|_H^2\leqslant 2\,\|f(t)\|_H\cdot\|u_\varepsilon(t)\|_H\leqslant\|f(t)\|_H^2+\|u_\varepsilon(t)\|_H^2$, it follows from (5) that $\{u_\varepsilon\}$ is bounded in $L^\infty(0, T; V)$. This and the boundedness of $P(a, \mathrm{grad})$ from V into H imply that $\{u'_\varepsilon\}$ is also bounded in $L^2(0, T; V')$. (Note that N is a bounded operator from V into V'.) Now we can apply the compactness theorem of J. P. Aubin (see [7, Chap. III]) to conclude that $\{u_\varepsilon\}$ contains a subsequence converging weak-star in

$L^{\infty}(0, T; V)$ and strongly in $L^2(0, T; H)$ to a solution u of (I.V.P.) such that $u' \in L^2(0, T; V')$. Hence it follows from an interpolation theorem that u belongs to $C([0, T]; H)$, from which we can easily deduce (c). The estimate (a) follows immediately from (5), and (b) is obvious. The proof of the uniqueness is omitted. Theorem 4 is proved by approximating $u_0 \in H$, $f(t) \in L^2(0, T; H)$ and $a(t) \in L^{\infty}(0, T; H)$ by the data satisfying the assumptions of Theorem 1, and using (c) of Theorem 2. Finally, Theorem 3 is a direct consequence of the following result, due to Faris-Lavine [3].

Theorem. *Let A be a symmetric operator and $S \geqslant 1$ be a self-adjoint operator on a Hilbert space X satisfying for $u \in D(S)$,*

(i) $\|Au\| \leqslant C_1 \|Su\|$,

(ii) $|(Au, Su) - (Su, Au)| \leqslant C_2 \|S^{1/2}u\|^2$,

with some positive constants C_1 and C_2 independent of u. Then A is essentially self-adjoint on any core of S.

In our case $A = iP(a, \text{grad})$ and $S = 1 + N$. The validity of (i) and (ii) is verified by a direct calculation using the definition and coerciveness of the Neumann problem. The symmetricity of $iP(a, \text{grad})$ follows easily by an integration by parts.

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