64. On a Nature of Convergence of Some Feynman Path Integrals. II

By Daisuke FUJIWARA
Department of Mathematics, University of Tokyo

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§1. **Introduction.** In the previous note [8], we reported that under the assumptions (V-I) and (V-II) below, the Feynman path integral with the Lagrangean

$$L(t, x, \dot{x}) = \frac{1}{2}|\dot{x}|^2 - V(t, x)$$

converges in a very strong topology if the time interval is short. In the present note, we shall discuss the convergence of the Feynman path integral in the case the time interval is longer.

The potential function $V(t, x)$ is assumed to satisfy the following two assumptions;

(V-I) *$V(t, x)$ is a real valued function of (t, x). For any fixed $t \in \mathbf{R}$, $V(t, x)$ is of class C^∞ in x. $V(t, x)$ is a measurable function of $(t, x) \in \mathbf{R} \times \mathbf{R}^n$.*

(V-II) *For any multi-index α with its length $|\alpha| \geq 2$, the measurable function $M_\alpha(t)$ defined by*

$$M_\alpha(t) = \sup_{x \in \mathbf{R}^n} \left| \left(\frac{\partial}{\partial x} \right)^\alpha V(t, x) \right| + \sup_{|x| \leq 1} |V(t, x)|$$

is essentially bounded on any compact set of $\mathbf{R}$.

We fix a large positive number K, say, $K = 100(n+100)$. We let $T = \infty$ if $\operatorname*{ess.sup}_{t \in \mathbf{R}} \sum_{2 \leq |\alpha| \leq K} M_\alpha(t) < \infty$. Otherwise, we let T be any fixed finite positive number. We shall discuss everything in the time interval $(-T, T)$.

Let $S(t, s, x, y)$ be the classical action along the classical orbit starting from y at time s and reaching x at time t. We can prove that there exists a positive constant $\delta_1(T)$ such that $S(t, s, x, y)$ is uniquely defined for any x and $y \in \mathbf{R}^n$ if $|t-s| \leq \delta_1(T)$. See, [6], [7], and [8]. For $N = 0, 1, 2, \cdots$, we shall consider the following integral transformation,

$$(1) \qquad E^{(N)}(\lambda, t, s)\varphi(x) = \left(\frac{-\lambda}{2\pi(t-s)} \right)^{n/2} \int_{\mathbf{R}^n} a^{(N)}(\lambda, t, s, x, y) e^{\lambda S(t, s, x, y)} \varphi(y) dy,$$

where $\lambda = \sqrt{-1}\hbar^{-1}$, $\hbar$ being a small positive parameter (the Planck's constant), and the amplitude function is defined by (3) and (11) of [8]. Note that $E^{(0)}(\lambda, t, s)$ is the integral transformation that was used by

Feynman [4] and [5]. See also Chazarain [2] and Kitada [9].

Let $[s, t]$ be an arbitrary subinterval of $(-T, T)$ and let

$$\Delta\,;\ s=s_0<s_1<s_2<\cdots<s_{L-1}<s_L=t \tag{2}$$

be an arbitrary subdivision of $[s, t]$. We denote

$$\delta(\Delta)=\max_{1\le j\le L}|s_j-s_{j-1}|. \tag{3}$$

Following Feynman, we consider the iterated operator $E^{(N)}(\Delta|\lambda, t, s)$ associated with the subdivision Δ;

$$E^{(N)}(\Delta|\lambda, t, s)=E^{(N)}(\lambda, t, s_{L-1})E^{(N)}(\lambda, s_{L-1}, s_{L-2})\cdots E^{(N)}(\lambda, s_1, s). \tag{4}$$

Its kernel function is

$$\begin{aligned} &I^{(N)}(\Delta|\lambda, t, s, x, y)\\ &=\prod_{j=1}^{L}\left(\frac{-\lambda}{2\pi(s_j-s_{j-1})}\right)^{n/2}\\ &\quad\times\int_{\mathrm{R}^n}\cdots\int_{\mathrm{R}^n}\prod_{j=1}^{L}a^{(N)}(\lambda, s_j, s_{j-1}, x^j, x^{j-1})\\ &\qquad\times\exp\lambda\sum_{j=1}^{L}S(s_j, s_{j-1}, x^j, x^{j-1})\prod_{j=1}^{L-1}dx^j, \end{aligned} \tag{5}$$

where $x^0=y$ and $x^L=x$. We shall prove that the limit

$$\lim_{\delta(\Delta)\to 0} I^{(N)}(\Delta|\lambda, t, s, x, y) \tag{6}$$

exists and equals to the kernel function of the fundamental solution for the Schrödinger equation

$$\begin{aligned} &\frac{\partial}{\lambda\partial t}u(t, x)-\left(2^{-1}\sum_{j=1}^{n}\left(\frac{\partial}{\lambda\partial x_j}\right)^2+V(t, x)\right)u(t, x)=0,\\ &u(s, x)=\varphi(x), \end{aligned} \tag{7}$$

if we further assume additional conditions (0-I)–(0-III) which will be stated in § 2. These conditions roughly mean that the terminal point (t, x) is not conjugate to the initial point (s, y) in the space time along any of classical orbits joining these points.

§ 2. Additional assumptions and main results. If $|t-s|\le\delta_1(T)$, we can write as

$$I^{(N)}(\Delta|\lambda, t, s, x, y)=\left(\frac{-\lambda}{2\pi(t-s)}\right)^{n/2}a^{(N)}(\Delta|\lambda, t, s, x, y)e^{\lambda S(t,s,x,y)} \tag{8}$$

with some amplitude function $a^{(N)}(\Delta|\lambda, t, s, x, y)$ which is an element $a^{(N)}(\Delta|\lambda, t, s)$ of $\mathscr{B}(\mathrm{R}^n_x\times\mathrm{R}^n_y)$ as a function of (x, y) with parameters Δ, λ, t and s. We proved in [8] that

$$\lim_{\delta(\Delta)\to 0} a^{(N)}(\Delta|\lambda, t, s, x, y)=k(\lambda, t, s, x, y) \tag{9}$$

exists in the function space $\mathscr{B}(\mathrm{R}^n_x\times\mathrm{R}^n_y)$ if $|t-s|\le\delta_1(T)$. (See Theorems 1 and 2 of [8]). Let

$$K(\lambda, t, s, x, y)=\left(\frac{-\lambda}{2\pi(t-s)}\right)^{n/2}k(\lambda, t, s, x, y)e^{\lambda S(t,s,x,y)} \tag{10}$$

and define the operator

$$U(\lambda, t, s)\varphi(x)=\int_{\mathrm{R}^n}K(\lambda, t, s, x, y)\varphi(y)dy. \tag{11}$$

Then, this is the fundamental solution for the Schrödinger equation (7).

We wish to treat the case that $|t-s|$ is larger. Let $[s, t] \subset (-T, T)$ be an arbitrary time interval. Let

(12) $$\Delta_1;\ s=t_0<t_1<\cdots<t_L=t$$

be any fixed subdivision such that

(13) $$\delta(\Delta_1)=\max_{1\leq j\leq L}|t_j-t_{j-1}|\leq\delta_1(T).$$

Then, by the evolution property of $U(\lambda, t, s)$, we define the fundamental solution $U(\lambda, t, s)$ by

(14) $$U(\lambda, t, s)=U(\lambda, t, t_{L-1})U(\lambda, t_{L-1}, t_{L-2})\cdots U(\lambda, t_1, s).$$

Let

(15) $$\Delta^i;\ t_i=s_{i,0}<s_{i,1}<\cdots<s_{i,k_k}=t_{i+1}$$

be an arbitrary subdivision of $[t_i, t_{i+1}]$. As a result of Theorems 1 and 2 of [8], we have

Theorem 1. *For non-negative integer m, there exists a positive constant $C_0(m, T)$ such that*

(16) $$\begin{aligned}&\|a^{(N)}(\Delta^i|\lambda, t_{i+1}, t_i)-k(\lambda, t_{i+1}, t_i)\|_m\\&\quad\leq C_0(m, T)\delta(\Delta^i)^{N+1}|\lambda|^{-N}|t_{i+1}-t_i|\exp C_0(m, T)|t_{i+1}-t_i|.\end{aligned}$$

In order to obtain more detailed description, we require further knowledge about geometry of the classical orbits. We shall treat the simplest case. We assume the following conditions;

(0-I) *There are g different classical orbits $\gamma_1, \gamma_2, \cdots, \gamma_g$ starting from y_0 at time s and reaching x_0 at time t.*

(0-II) *x_0 is not conjugate to y_0 along any of these classical orbits.* It is well known that if these two conditions hold then they also hold for any pair (x, y) of points in some neighbourhood of $(x_0, y_0)\in \mathbf{R}\times\mathbf{R}^n$. (See [11].)

To state the third assumption (0-III), we need notations: We fix the subdivision Δ_1 as (12). Take any sequence of points $y=x^0, x^1, \cdots, x^{L-1}$, $x^L=x$ in $\mathbf{R}^n$. Let us denote by $\gamma^{j,j-1}$ the classical orbit joining (t_{j-1}, x^{j-1}) to (t_j, x^j) in the space time $\mathbf{R}\times\mathbf{R}^n$. If we connect all these, we obtain a broken classical orbit $\gamma(x, x^{L-1}, \cdots, x^1, y)$ joining (t, x) to (s, y) in $\mathbf{R}\times\mathbf{R}^n$. We shall denote by η^{j-1} and ξ^j the initial and the terminal, respectively, momenta of the classical orbit $\gamma^{j,j-1}$. If the broken orbit $\gamma(x, x^{L-1}, \cdots, x^1, y)$ differs from any of the smooth classical orbits $\gamma_1, \gamma_2, \cdots, \gamma_g$, then the momentum along the broken orbit $\gamma(x, x^{L-1}, \cdots, x^1, y)$ is discontinuous at some of the points (t_j, x^j), $j=1, 2, \cdots, L-1$, that is, $\xi^j\neq\eta^j$ for some $j=1, 2, \cdots, L-1$. We require a little stronger property;

(0-III) *Let $\zeta^j=\xi^j-\eta^j$. Then,* $\lim\sum_{j=1}^{L-1}|\zeta^j|^2=\infty$ *as* $\sum_{j=1}^{L-1}|x^j|^2\to\infty$. *And there exist positive constants γ_{11} and γ_{12} such that*

(17) $$\left|\det \frac{\partial(\zeta^1\cdots\zeta^{L-1})}{\partial(x^1\cdots x^{L-1})}\right| \geq \gamma_{12} \qquad if \sum_{j=1}^{L-1} |x^j|^2 \geq \gamma_{11}.$$

Theorem 2. *In addition to* (V-I) *and* (V-II), *we assume* (0-I)–(0-III). *Then, the kernel function* $K(\lambda, t, s, x, y)$ *of* $U(\lambda, t, s)$ *is of the form*

(18) $$\begin{aligned} K(\lambda, t, s, x, y) &= \left(\frac{-\lambda}{2\pi(t-s)}\right)^{n/2} \\ &\times \sum_{j=1}^{g} k_j(\lambda, t, s, x, y) \exp\left\{-\frac{\pi}{2} i \operatorname{Ind} \gamma_j + \lambda S_j(t, s, x, y)\right\}, \end{aligned}$$

where $S_j(t, s, x, y)$ *is the classical action along the classical orbit* γ_j *and* $\operatorname{Ind} \gamma_j$ *is the Morse-Keller-Maslov index of the orbit* γ_j. *The function* $k_j(\lambda, t, s, x, y)$ *is a smooth function in some neighbourhood of* (t, s, x_0, y_0).

By the same reason, we can prove that the kernel function $I^{(N)}(\Delta|\lambda, t, s, x, y)$ of $E^{(N)}(\Delta|\lambda, t, s)$ has the form

(19) $$\begin{aligned} I^{(N)}(\Delta|\lambda, t, s, x, y) &= \left(\frac{-\lambda}{2\pi(t-s)}\right)^{n/2} \\ &\times \sum_{j=1}^{g} b_j^{(N)}(\Delta|\lambda, t, s, x, y) \exp\left\{-\frac{\pi}{2} i \operatorname{Ind} \gamma_j + \lambda S(t, s, x, y)\right\}, \end{aligned}$$

where we put $\Delta = \Delta^1 \cup \Delta^2 \cup \cdots \cup \Delta^L$, the refinement of Δ_1 of (12). We wish to discuss the limit of $I^{(N)}(\Delta|\lambda, t, s, x, y)$ as $\delta(\Delta) \to 0$.

Theorem 3. *In addition to* (V-I) *and* (V-II), *we assume* (0-I)–(0-III). *Then, for a small neighbourhood* F *of* (x_0, y_0) *in* $\mathbf{R}^n \times \mathbf{R}^n$, *there exists a positive constant* C *such that*

(20) $$|k_j(\lambda, t, s, x, y) - b_j^{(N)}(\Delta|\lambda, t, s, x, y)| \leq C|\lambda|^{-N} \delta(\Delta)^{N+1}$$

holds for any $(x, y) \in F$ *and for* $j = 1, 2, 3, \cdots, g$. *The constant* C *is independent of subdivision* Δ *and of* λ *if* $|\lambda|$ *is bounded away from* 0.

This enables us to discuss the behaviour of $K(\lambda, t, s, x, y)$ as $|\lambda| \to \infty$ (the quasi-classical limit). Following Yajima's calculation in [12], we have

Theorem 4. *Assume* (0-I)–(0-III) *as well as* (V-I) *and* (V-II). *Then, we have the estimate*

(21) $$\left|\left(\frac{1}{t-s}\right)^{n/2} k_j(\lambda, t, s, x, y) - \left|\frac{dy \wedge d\eta}{dy \wedge dx(\gamma_j(t))}\right|^{1/2}\right| \leq C|\lambda|^{-1},$$

where $dy \wedge d\eta$ *is the canonical* $2n$ *form in the phase space* $\mathbf{R}^n \times \mathbf{R}^n$ *and* $dy \wedge dx(\gamma_j(t))$ *denotes the volume element of the space* $\mathbf{R}^n \times \mathbf{R}^n$ *which is the direct product of the initial and the terminal configuration space along the orbit* γ_j, $j = 1, 2, 3, \cdots, g$.

§ 3. Sketch of the proof of Theorem 2. We have

(22) $$\begin{aligned} &K(\lambda, t, s, x, y) \\ &= \prod_{j=1}^{L} \left(\frac{-\lambda}{2\pi(t_j - t_{j-1})}\right)^{n/2} \end{aligned}$$

$$\int_{\mathbf{R}^n} \cdots \int_{\mathbf{R}^n} \prod_{j=1}^{L} k(\lambda, t_j, t_{j-1}, x^j, x^{j-1}) \exp \lambda S(t_j, t_{j-1}, x^j, x^{j-1}) \prod_{j=1}^{L-1} dx^j.$$

We can apply the stationary phase method to this integral (22) and obtain Theorem 2 (see Fedoryuk [3] or Asada-Fujiwara [1]), since (0-I)–(0-III) hold.

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