60. *A Generalization of Cauchy-Riemann Equations on a Riemannian Symmetric Space and the H^p Space Theory*

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We consider a generalization of Cauchy-Riemann equations in a Riemannian symmetric space and we extend the theory of H^p spaces by using this generalization.

We list some examples of generalizations of Cauchy-Riemann equations.

(a) E. M. Stein and C. Weiss [5] have defined Cauchy-Riemann equations in the n-dimensional Euclidean space in the following setting:

$$(1) \qquad \sum_{i=1}^{n} \partial u_i/\partial x_i = 0, \qquad \partial u_i/\partial x_j = \partial u_j/\partial x_i.$$

They obtained that each u_i is harmonic and that $|u|^p$ is subharmonic if $p \geqq (n-2)/(n-1)$ where $|u| = (|u_1^2| + \cdots + |u_n|^2)^{1/2}$.

(b) C. Fefferman and E. M. Stein [3] directly generalized the system (1) in the n-dimensional Euclidean space.

(c) The system (1) was extended to a compact Lie group by R. R. Coifman and G. Weiss [2].

(d) Let M be a Riemannian manifold and let d be the exterior differential operator on M and δ the codifferential operator. Then the deRham-Hodge equations $d\omega = \delta\omega = 0$ can be considered as a generalization of Cauchy-Riemann equations.

(e) The "spinor" system given by the Dirac operator on a spin manifold is a generalization of Cauchy-Riemann equations (see M. F. Atiyah [1]).

In this paper an extension of all these examples in a Riemannian symmetric space will be given as follows:

(i) We consider a homogeneous vector bundle over a Riemannian symmetric space such that its fiber is a Clifford algebra.

(ii) Next we consider C^∞ cross sections on such a homogeneous vector bundle in Lie algebra level (see Definition 1).

(iii) A generalization of Cauchy-Riemann equations is given by a certain differential operator d and its dual δ operating on such C^∞ cross sections, that is,

$$(2) \qquad d\omega = \delta\omega = 0$$

(see Definition 2). The examples (a), (b), (c) and (d) will arise when the

Clifford algebra is an exterior algebra. The example (e) will arise when the Clifford algebra is a "spinor" algebra.

In Theorems 1 and 2 we shall see that a solution ω of the system (2) is harmonic and $|\omega|^p$ is subharmonic if $p \geqq (n-2)/(n-1)$ in a certain sense, and using these properties we can extend results of the H^p space theory (the Poisson representation theorem, F. and M. Riesz's theorem, etc.).

Let $(\mathfrak{G}, \sigma)$ be an effective orthogonal symmetric Lie algebra where $\mathfrak{G}$ is a Lie algebra over R and σ is an involutive automorphism of $\mathfrak{G}$. In this paper we assume that $(\mathfrak{G}, \sigma)$ is of the noncompact type, of the compact type or of the Euclidean type. Let $\mathfrak{G}=\mathfrak{K}+\mathfrak{P}$ be the decomposition of $\mathfrak{G}$ into the eigenspaces of σ for the eigenvalue $+1$ and -1, respectively. Let (G, K) be a Riemannian symmetric pair associated with $(\mathfrak{G}, \sigma)$. Let m and n denote the dimensions of $\mathfrak{K}$ and $\mathfrak{P}$, respectively. To avoid triviality we assume that $n \geqq 2$. We denote by B the Killing form of $\mathfrak{G}$. We choose once and for all an orthogonal basis $Z_1, \cdots, Z_m, X_1, \cdots, X_n$ of $\mathfrak{G}$ with respect to the Killing form B such that $Z_j \in \mathfrak{K}$, $j=1, \cdots, m$ and $X_i \in \mathfrak{P}$, $i=1, \cdots, n$. Moreover, we suppose that

$$B(Z_j, Z_j) = -1, \qquad j=1, \cdots, m$$

and

(i) if $B(X_i, X_i)>0$, $i=1, \cdots, n$ then $B(X_i, X_i)=1$,

(ii) if $B(X_i, X_i)<0$, $i=1, \cdots, n$ then $B(X_i, X_i)=1$,

(iii) if $B(X_i, X_i)=0$, $i=1, \cdots, n$ then $\{X_i\}$ is orthonormal with respect to an inner product which is invariant under $Ad(k)$ $(k \in K)$.

We may consider elements of $\mathfrak{G}$ as left invariant differential operators on G. We denote by $e_1, \cdots, e_n$ a basis of the vector space $\mathfrak{P}$ corresponding to $X_1, X_2, \cdots, X_n$. We denote by $C_+(\mathfrak{P})$, $C_-(\mathfrak{P})$ and $C_0(\mathfrak{P})$ the Clifford algebras defined by symmetric bilinear forms $(e_i | e_j)_+ = \delta_{ij}$, $(e_i | e_j)_- = -\delta_{ij}$ and $(e_i | e_j)_0 = 0$, $i, j=1, \cdots, n$, respectively. We denote by $\tilde{C}_+(\mathfrak{P})$, $\tilde{C}_-(\mathfrak{P})$ and $\tilde{C}_0(\mathfrak{P})$ the complexifications of $C_+(\mathfrak{P})$, $C_-(\mathfrak{P})$ and $C_0(\mathfrak{P})$, respectively. $C(\mathfrak{P})$ denotes any one of $C_+(\mathfrak{P})$, $C_-(\mathfrak{P})$ and $C_0(\mathfrak{P})$, and $\tilde{C}(\mathfrak{P})$ denotes its cemplexifications. We denote by $C^\infty(G; C(\mathfrak{P}))$ and $C^\infty(G; \tilde{C}(\mathfrak{P}))$ the spaces of all C^∞ functions on G with values in $C(\mathfrak{P})$ and $\tilde{C}(\mathfrak{P})$, respectively. Let $\{c_{ij}{}^k\}$ be a set of constants such that

$$ad(Z_k)X_j = \sum_{i=1}^n c_{ij}{}^k X_i, \qquad k=1, \cdots, m, \quad j=1, \cdots, n,$$

where ad is the adjoint representation of $\mathfrak{G}$. We define a linear mapping $\tau(Z): \tilde{C}(\mathfrak{P}) \to \tilde{C}(\mathfrak{P})$, $Z \in \mathfrak{K}$ as follows:

(i) When $\tilde{C}(\mathfrak{P}) = \tilde{C}_+(\mathfrak{P})$, we set $\tau(Z_k)$ = *left Clifford multiplication by*

$$(1/4) \sum_{i,j} c_{ij}{}^k e_i e_j, \qquad k=1, \cdots, m.$$

(ii) When $\tilde{C}(\mathfrak{P}) = \tilde{C}_-(\mathfrak{P})$, we set $\tau(Z_k)$ = *left Clifford multiplication by*

$$-(1/4)\sum_{i,j} c_{ij}{}^k e_i e_j, \qquad k=1,\cdots,m.$$

(iii) When $\tilde{C}(\mathfrak{P})=\tilde{C}_0(\mathfrak{P})$, we set

$$\tau(Z_k)=\sum_{i,j} c_{ij}{}^k e_i \iota(e_j), \qquad k=1,\cdots,m.$$

A mapping $\iota(e_j): \tilde{C}(\mathfrak{P})\to\tilde{C}(\mathfrak{P})$, $j=1,\cdots,n$, is as follows: If $\xi\in\tilde{C}(\mathfrak{P})$ has a form $\xi=\xi_1+e_j\xi_2$ where all terms of ξ_1 and ξ_2 do not contain e_j then we set $\iota(e_j)\xi=\xi_2$.

Definition 1. We put

$$C_\tau^\infty(G;\tilde{C}(\mathfrak{P}))=\{\omega\in C^\infty(G;\tilde{C}(\mathfrak{P})): Z\omega=\tau(-Z)\omega \text{ for all } Z\in\mathfrak{K}\}$$

and

$$C_\tau^\infty(G;C(\mathfrak{P}))=\{\omega\in C^\infty(G;C(\mathfrak{P})): Z\omega=\tau(-Z)\omega \text{ for all } Z\in\mathfrak{K}\}.$$

We set

$$(\omega,\xi)=\int_G \langle\omega(g),\xi(g)\rangle dg$$

for suitable elements $\omega,\xi\in C^\infty(G;\tilde{C}(\mathfrak{P}))$, where the inner product $\langle\ ,\ \rangle$ is a natural inner product in $\tilde{C}(\mathfrak{P})$.

Definition 2. We define an operator

$$d: C_\tau^\infty(G;\tilde{C}(\mathfrak{P}))\to C_\tau^\infty(G;\tilde{C}(\mathfrak{P}))$$

by

$$d\omega(g)=\sum_{i=1}^n e_i X_i\omega(g)$$

and an operator $\delta: C_\tau^\infty(G;\tilde{C}(\mathfrak{P}))\to C_\tau^\infty(G;\tilde{C}(\mathfrak{P}))$ to be the formally adjoint operator of d with respect to the inner product $(\ ,\ )$.

We now come to the definition of a generalization of Cauchy-Riemann equations. We define it by equations

(3) $$d\omega=\delta\omega=0$$

for $\omega\in C_\tau^\infty(G;\tilde{C}(\mathfrak{P}))$.

Example 1. We put $G=R^n$, the n-dimensional Euclidean space and $K=\{0\}$. Then (G,K) is a Riemannian symmetric pair of the Euclidean type. The Clifford algebra $C_0(R^n)$ is the exterior algebra of R^n. For a 1-form $\omega\in C^\infty(R^n;C_0(R^n))$, the system (3) is the system (1) of Cauchy-Riemann equations in the sense of E. M. Stein and G. Weiss [5]. In general, for any form $\omega\in C^\infty(R^n;C_0(R^n))$, the system (3) is a generalization of Cauchy-Riemann equations in the sense of C. Fefferman and E. M. Stein [3].

Example 2. Let (G,K) be a Riemannian symmetric pair as before and let $M=G/K$ be the Riemannian symmetric space. We denote by $\wedge^* T(M)$ the exterior algebra generated by the dual of the tangent bundle over M. The bundle $\wedge^* T(M)$ is a homogeneous vector bundle over M associated with the adjoint representation $(\mathrm{Ad}\,(k), C_0(\mathfrak{P}))$ of K. Then the space $\Gamma^\infty(\wedge^* T(M))$ of all C^∞ cross sections of $\wedge^* T(M)$ is isomorphic to the space

$$C_{Ad}^\infty(G;C_0(\mathfrak{P}))=\{\omega\in C^\infty(G;C_0(\mathfrak{P})): \omega(gk)=Ad(k^{-1})\omega(g),\ k\in K,\ g\in G\}$$

and this may be considered in Lie algebra level as the space

$$C^{\infty}_{ad}(G\,;C_0(\mathfrak{P}))=\{\omega\in C^{\infty}(G\,;C_0(\mathfrak{P})) : Z\omega=ad(-Z)\omega,\ Z\in\mathfrak{K}\}.$$

Hence a solution of equations (3) corresponds to a harmonic form in the sense of deRham-Hodge. If G is a semisimple compact connected Lie group and G^* is the subgroup $\{(x,x) : x\in G\}$ of the product group $G\times G$, then $(G\times G, G^*)$ is a Riemannian symmetric pair of the compact type and G can be regarded as the Riemannian symmetric space $G\times G/G^*$. In this case, for a 1-form $\omega\in C^{\infty}_{Ad}(G\times G\times R_+\,;C_0(\mathfrak{P}))$, where R_+ is the positive half line, the system (3) corresponds to the system of R. R. Coifman and G. Weiss [2].

Example 3. Let V be a real vector space with even dimension $n=2l$. Let Q_j be the transformation of the complexified Clifford algebra $\tilde{C}_-(V)$ given by right Clifford multiplication by $\sqrt{-1}e_{2j-1}e_{2j}$, $j=1,\cdots,l$. We define

$$S(V)=\{\omega\in\tilde{C}_-(V) : Q_j\omega=-\omega,\ j=1,\cdots,l\}.$$

We put $G=R^n$ (n even) and $K=\{0\}$. Then, for $\omega\in C^{\infty}(R^n\,;S(R^n))\subset C^{\infty}(R^n\,;\tilde{C}_-(R^n))$, a solution of the system (3) is a harmonic spinor for the Dirac operator.

Let (G,K) be a Riemannian symmetric pair associated with an effective orthogonal symmetric Lie algebra $(\mathfrak{G},\sigma)$ of the noncompact type and let M be the Riemannian symmetric space G/K with even dimension $n=2l$. $S(M)$ denotes a homogeneous vector bundle associated with a representation $(\widetilde{Ad}(k), S(\mathfrak{P}))$ of K where $\widetilde{Ad}(k)$ is a lifting of $Ad(k)$ ($k\in K$) to $Spin(\mathfrak{P})$. Then, for

$$\omega\in C^{\infty}_{\widetilde{Ad}}(G\,;S(\mathfrak{P}))\subset C^{\infty}_{\widetilde{Ad}}(G\,;\tilde{C}_-(\mathfrak{P})),$$

a solution of the system (3) corresponds to a harmonic spinor for the Dirac operator on $S(M)$.

Theorem 1 (Harmonicity). *Suppose that ω is a solution of the system* (3) *in $C^{\infty}_{\tau}(G\,;\tilde{C}(\mathfrak{P}))$.*

(i) *When $\mathfrak{G}$ is of the noncompact type, we have*

$$\left(\sum_{j=1}^{n}X_j^2-2\sum_{k=1}^{m}Z_k^2\right)\omega=0 \qquad if\ \tilde{C}(\mathfrak{P})=\tilde{C}_+(\mathfrak{P})\ \text{or}\ \tilde{C}_-(\mathfrak{P})$$

and

$$\left(\sum_{j=1}^{n}X_j^2-\sum_{k=1}^{m}Z_k^2\right)\omega=0 \qquad if\ \tilde{C}(\mathfrak{P})=\tilde{C}_0(\mathfrak{P}).$$

(ii) *When $\mathfrak{G}$ is of the compact type, we have*

$$\left(\sum_{j=1}^{n}X_j^2+2\sum_{k=1}^{m}Z_k^2\right)\omega=0 \qquad if\ \tilde{C}(\mathfrak{P})=\tilde{C}_+(\mathfrak{P})\ or\ \tilde{C}_-(\mathfrak{P})$$

and

$$\left(\sum_{j=1}^{n}X_j^2+\sum_{k=1}^{m}Z_k^2\right)\omega=0 \qquad if\ \tilde{C}(\mathfrak{P})=\tilde{C}_0(\mathfrak{P}).$$

(iii) *When $\mathfrak{G}$ is of the Euclidean type, we have*

$$\left(\sum_{j=1}^{n}X_j^2\right)\omega=0.$$

Theorem 2 (Subharmonicity). *Suppose that ω is a solution of the system* (3) *in $C_\tau^\infty(G; C(\mathfrak{P}))$ and $p \geqq (n-2)/(n-1)$.*

(i) *When $\mathfrak{G}$ is of the compact type we have*

$$\left(\sum_{j=1}^{n} X_j^2 + 2\sum_{k=1}^{m} Z_k^2\right)|\omega|^p \geqq 0 \qquad \text{if } C(\mathfrak{P}) = C_+(\mathfrak{P}) \text{ or } C_-(\mathfrak{P})$$

and

$$\left(\sum_{j=1}^{n} X_j^2 + \sum_{k=1}^{m} Z_k^2\right)|\omega|^p \geqq 0 \qquad \text{if } C(\mathfrak{P}) = C_0(\mathfrak{P}).$$

(ii) *When $\mathfrak{G}$ is of the noncompact type or of the Euclidean type we have*

$$\left(\sum_{j=1}^{n} X_j^2\right)|\omega|^p \geqq 0.$$

Next we will present an extension of H^p spaces. Let R be the real line and let R_+ be the positive half line. We put $G_+ = G \times R_+$ and $\mathfrak{P}' = \mathfrak{P} + R$. We define H^p spaces $(p>0)$ given by

$$H^p = \left\{\omega \in C_\tau^\infty(G_+; C(\mathfrak{P}')) : d\omega = \delta\omega = 0, \right.$$
$$\left. \|\omega\|_{H^p} = \sup_{t>0}\left(\int_G |\omega(x,t)|^p\, dx\right)^{1/p} < \infty \right\}.$$

We can construct a Poisson semigroup $\{P_t\}_{t>0}$ defined on $L^p(G)$, $1 \leqq p \leqq \infty$, by the Laplacian $\sum_{j=1}^n X_j^2 + c\sum_{k=1}^m Z_k^2$ where

$$c = \begin{cases} 2 & \text{if } C(\mathfrak{P}') = C_+(\mathfrak{P}') \text{ or } C_-(\mathfrak{P}') \\ 1 & \text{if } C(\mathfrak{P}') = C_0(\mathfrak{P}') \end{cases}$$

(see K. Saka [4]). The Poisson semigroup $\{P_t\}_{t>0}$ can be also defined on the space $L^p(G; C(\mathfrak{P}'))$ of all L^p-functions on G with values in $C(\mathfrak{P}')$.

A following theorem is an extension of the representation theorem and F. and M. Riesz's theorem. The theorem can be proved from Theorems 1 and 2 (see K. Saka [4]).

Theorem 3. *Assume that $1 \leqq p \leqq \infty$.*

(i) *Suppose that $\mathfrak{G}$ is of the compact type and $\omega \in H^p$. Then ω can be represented as a Poisson integral $P_t f$ of a certain element f in $L^p(G; C(\mathfrak{P}'))$.*

(ii) *Suppose that $\mathfrak{G}$ is of the noncompact type or of the Euclidean type and $\omega \in H^p$ satisfies the relation*

(4) $\qquad \omega(gk, t) = \omega(g, t) \qquad$ *for $k \in K$, $t \in R_+$ and $g \in G$.*

Then ω can be represented as a Poisson integral $P_t f$ of a certain element f in $L^p(G; C(\mathfrak{P}'))$.

A following characterization theorem can be derived from Theorem 2 (see K. Saka [4]).

Theorem 4. *Assume that $(n-1)/n < p < \infty$ and that ω is a solution of the system* (3) *in $C_\tau^\infty(G_+; C(\mathfrak{P}'))$.*

(i) *Either suppose that $\mathfrak{G}$ is of the compact type, or*

(ii) *suppose that $\mathfrak{G}$ is of the noncompact type or of the Euclidean type and ω satisfies the relation* (4). *Then $\omega \in H^p$ if and only if*

$$\sup_{t>0} |\omega(g, t)| = \omega^+(g) \in L^p(G).$$

In this case, there are positive constants C and C' such that

$$\|\omega\|_{H^p} \leqq C \|\omega^+\|_p \leqq C' \|\omega\|_{H^p}.$$

Details of these results will appear elsewhere.

References

[1] M. F. Atiyah: Classical groups and classical differential operators on manifolds. C.I.M.E. III, Differential Operators on Manifold, Coordinatore, E. Vesentini, pp. 5–48 (1975).

[2] R. R. Coifman and G. Weiss: Invariant systems of conjugate harmonic functions associated with compact Lie group. Studia Math., **44**, 301–308 (1972).

[3] C. Fefferman and E. M. Stein: H^p spaces of several variables. Acta Math., **129**, 137–193 (1972).

[4] K. Saka: The representation theorem and the H^p space theory associated with semigroups on Lie groups. Tôhoku Math. J., **30**, 131–151 (1978).

[5] E. M. Stein and G. Weiss: On the theory of harmonic functions of several variables. I. Acta Math., **103**, 25–62 (1960).