59. *Isomorphism Criterion and Structure Group Description for $\mathfrak{N}$-Semigroups*

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0. Introduction. A commutative cancellative archimedean semigroup without idempotents is called an $\mathfrak{N}$-semigroup. In this paper, necessary and sufficient conditions are given for two $\mathfrak{N}$-semigroups to be isomorphic and the structure groups of an $\mathfrak{N}$-semigroup are completely described. M. Sasaki did some related work in [1], but the results given here are simpler. In [2], T. Tamura obtained an isomorphism criterion from a different point of view.

1. Preliminaries. Let S be any $\mathfrak{N}$-semigroup and let $a\in S$. Define a group-congruence ρ_a on S by $x\rho_a y$ if and only if $a^m x=a^n y$ for some $m,n\in Z_+$ (the positive integers). The group $G_a=S/\rho_a$ is called the structure group of S with respect to a. Structure group products will be denoted by "$*$" in this paper. Let $p\in S$. If $p\notin aS$, p is called a prime (relative to a). Every $x\in S$ has a unique representation $x=a^k p$ where $k\in Z_+^0$ ($a^0 p$ means p) and $p\in S$ is a prime. By the fundamental structure theorems for $\mathfrak{N}$-semigroups [2], we may assume $S=(G;I)=(G;\varphi)$, that is, $S=\{(x,\xi): x\in Z_+^0, \xi\in G\}$ where $(x,\xi)(y,\eta)=(x+y+I(\xi,\eta), \xi*\eta)$ and $I(\xi,\eta)=\varphi(\xi)+\varphi(\eta)-\varphi(\xi*\eta)$ for all $\xi,\eta\in G$. Let $(m,\alpha)\in S$. The structure group $G_{(m,\alpha)}=S/\rho$ is of major importance in this paper. Observe that $G_{(m,\alpha)}=\{\overline{(x,\xi)}: (x,\xi)$ is prime relative to $(m,\alpha)\}$. For a more thorough review of $\mathfrak{N}$-semigroups, see [2].

2. Isomorphism criterion. Theorem 2.1. *Let* $S=(G;I)=(G;\varphi)$ *and* $\hat{S}=(\hat{G};\hat{I})=(\hat{G};\hat{\varphi})$. *Then S is isomorphic to $\hat{S}$ if and only if there exists* $(m,\alpha)\in S$ *such that*

(2.1.1) $G_{(m,\alpha)}$ *is isomorphic to* $\hat{G}$ *and*

(2.1.2) $\hat{\varphi}(\hat{\xi})+\hat{\varphi}(\hat{\eta})-\hat{\varphi}(\hat{\xi}*\hat{\eta})=(x+\varphi(\xi)+y+\varphi(\eta)-(z+\varphi(\gamma)))/(m+\varphi(\alpha))$

holds for all $\hat{\xi},\hat{\eta}\in\hat{G}$ *where* (x,ξ), (y,η), *and* (z,γ) *are the unique primes in S relative to* (m,α) *such that the isomorphism in* (2.1.1) *carries* $\overline{(x,\xi)}, \overline{(y,\eta)}$, *and* $\overline{(z,\gamma)}$ *to* $\hat{\xi},\hat{\eta}$, *and* $\hat{\xi}*\hat{\eta}$ *respectively.*

Proof. *Necessity.* Assume $f: S\to\hat{S}$ is the isomorphism and let $f(m,\alpha)=(0,\hat{\varepsilon})$. Define $\iota: \hat{G}_{(0,\hat{\varepsilon})}\to\hat{G}$ by $\iota\overline{(0,\hat{\xi})}=\hat{\xi}$ and $\bar{f}: G_{(m,\alpha)}\to\hat{G}_{(0,\hat{\varepsilon})}$ by $\bar{f}\overline{(x,\xi)}=\overline{f(x,\xi)}$. Then $\iota\circ\bar{f}$ is an isomorphism of $G_{(m,\alpha)}$ onto $\hat{G}$. To prove (2.1.2), let $\hat{\xi},\hat{\eta}\in\hat{G}$ and let (x,ξ), (y,η), and (z,γ) be the primes relative to (m,α) such that $(\iota\circ\bar{f})\overline{(x,\xi)}=\hat{\xi}$, $(\iota\circ\bar{f})\overline{(y,\eta)}=\hat{\eta}$, and $(\iota\circ\bar{f})\overline{(z,\gamma)}=\hat{\xi}*\hat{\eta}$. Then $f(x,\xi)=(0,\hat{\xi})$, $f(y,\eta)=(0,\hat{\eta})$, and $f(z,\gamma)=(0,\hat{\xi}*\hat{\eta})$. Define a map $\hat{\varphi}': \hat{G} \to R_+$ (the positive reals) by $\hat{\varphi}'(\hat{\xi}) = (x + \varphi(\xi))/(m + \varphi(\alpha))$ where $(x, \xi) \in S$ such that $f(x, \xi) = (0, \hat{\xi})$. It can be shown that $\hat{I}(\hat{\xi}, \hat{\eta}) = \hat{\varphi}'(\hat{\xi}) + \hat{\varphi}'(\hat{\eta}) - \hat{\varphi}'(\hat{\xi} * \hat{\eta})$. From this fact, (2.1.2) follows easily.

Sufficiency. Let $g : G_{(m,\alpha)} \to \hat{G}$ be the isomorphism given in (2.1.1). To simplify the notation, when $(z, \gamma) \in S$ is a prime relative to (m, α), let $\hat{\gamma}_z$ denote the element $g\overline{(z, \gamma)}$. Recall that if $(x, \xi) \in S$, there is a unique representation $(x, \xi) = (m, \alpha)^k (z, \gamma)$ where $k \in Z_+^0$ and (z, γ) is a prime. Using this fact, define a map $f : S \to \hat{S}$ by $f(x, \xi) = (k, \hat{\gamma}_z)$. It is routine to show that f is well defined, one-to-one, and onto. Let (x, ξ), $(y, \eta) \in S$ and suppose $(x, \xi) = (m, \alpha)^j (w, \tau)$, $(y, \eta) = (m, \alpha)^n (v, \beta)$, and $(x + y + I(\xi, \eta), \xi * \eta) = (m, \alpha)^k (z, \gamma)$ where $j, n, k \in Z_+^0$ and (w, τ), (v, β), (z, γ) are primes. By using the fact that $(w + \varphi(\tau) + (v + \varphi(s))) - (z + \varphi(\gamma)) = (-j - n + k)(m + \varphi(\alpha))$, it can be shown that $f(x, \xi) \cdot f(y, \eta) = f((x, \xi)(y, \eta))$.

3. The structure groups. Let $S = (G; I) = (G; \varphi)$ where ε is the identity of G.

Lemma 3.1. *Let $(x, \xi) \in S$. Then (x, ξ) is prime relative to (m, α) if and only if $0 \leq x < m + I(\alpha, \alpha^{-1} * \xi)$.*

Lemma 3.2. *Let $(x, \xi) \in S$ and let (z, γ) be the unique prime in S relative to (m, α) such that $\overline{(x, \xi)} = \overline{(z, \gamma)}$. Then*

$$(z, \gamma) = \left(x - km - \sum_{i=1}^{k} I(\alpha, \alpha^{-i} * \xi), \alpha^{-k} * \xi \right)$$

where k is the unique non-negative integer satisfying

$$km + \sum_{i=1}^{k} I(\alpha, \alpha^{-i} * \xi) \leq x < (k+1)m + \sum_{i=1}^{k+1} I(\alpha, \alpha^{-i} * \xi).$$

In the following theorem, $\langle \alpha \rangle$ denotes the cyclic subgroup of G generated by α and the product in $G/\langle \alpha \rangle$ is denoted by "$*$".

Theorem 3.3. *Define a map $h : G_{(m,\alpha)} \to G/\langle \alpha \rangle$ by $h\overline{(x, \xi)} = \bar{\xi}$ where $\bar{\xi}$ denotes the congruence class* mod $\langle \alpha \rangle$ *containing ξ.*

(3.3.1) *The map h is a homomorphism from $G_{(m,\alpha)}$ to $G/\langle \alpha \rangle$.*

(3.3.2) Ker $(h) = \{\overline{(x, \xi)} \in G_{(m,\alpha)} : \xi = \alpha^n$ *for some* $n \in Z\}$.

(3.3.3) Ker $(h) = \langle \overline{(0, \varepsilon)} \rangle$, i.e., *the cyclic subgroup of $G_{(m,\alpha)}$ generated by $\overline{(0, \varepsilon)}$.*

Consequently, $G_{(m,\alpha)}$ is an abelian extension of $\langle \overline{(0, \varepsilon)} \rangle$ by $G/\langle \alpha \rangle$.

Proof. It is easy to verify (3.3.1) and (3.3.2). For (3.3.3), we first prove that $\langle \overline{(0, \varepsilon)} \rangle \subseteq$ Ker (h). We only need to show that $\overline{(0, \varepsilon)}^n \in$ Ker (h) for $n < 0$. Let $n = -k$, so $k > 0$. If $m \in Z_+$, we have $\overline{(0, \varepsilon)}^{-k} = \overline{((0, \varepsilon)^{-1})^k} = \overline{(m-1, \alpha)^k} = \overline{\left(k(m-1) + \sum_{i=1}^{k-1} I(\alpha, \alpha^i), \alpha^k \right)} \in$ Ker (h). If $m = 0$, let $j - 1$ be the first positive integer such that $I(\alpha, \alpha^{j-1}) \neq 0$. We then have

$$\begin{aligned} \overline{(0, \varepsilon)}^{-k} &= \overline{((0, \varepsilon)^{-1})^k} = \overline{(I(\alpha, \alpha^{j-1}) - 1, \alpha^j)^k} \\ &= \overline{\left(k(I(\alpha, \alpha^{j-1}) - 1) + \sum_{i=1}^{k-1} I(\alpha^j, \alpha^{ji}), \alpha^{jk} \right)} \in \text{Ker}\,(h). \end{aligned}$$

Next, we prove that Ker $(h)\subseteq\langle\overline{(0,\varepsilon)}\rangle$. Let $\overline{(x,\alpha^n)}\in$ Ker (h). Suppose $n>0$. Since $\overline{(x,\alpha^n)}*\overline{(0,\alpha^{-n})}=\overline{(x+I(\alpha^n,\alpha^{-n}),\varepsilon)}=\overline{(0,\varepsilon)}^{(x+I(\alpha^n,\alpha^{-n})+1)}$, we have $\overline{(x,\alpha^n)}=\overline{(0,\varepsilon)}^{(x+I(\alpha^n,\alpha^{-n})+1)}*\overline{(0,\alpha^{-n})}^{-1}$. It can be shown that $\overline{(0,\alpha^{-n})}=\overline{(0,\varepsilon)}^{(nm+[\sum_{i=1}^n(I(\alpha,\alpha^i)]+1)}$, hence $\overline{(0,\alpha^{-n})}^{-1}=\overline{(0,\varepsilon)}^{(-nm-[\sum_{i=1}^n I(\alpha,\alpha^{-i})]-1)}$. It follows that $\overline{(x,\alpha^n)}=\overline{(0,\varepsilon)}^{(x+I(\alpha^n,\alpha^{-n})-nm-\sum_{i=1}^n I(\alpha,\alpha^{-i}))}$. Now suppose $n=-k\leq 0$, so $k\geq 0$. If $x=0$, then $\overline{(0,\alpha^{-k})}=\overline{(0,\varepsilon)}^{(km+[\sum_{i=1}^k I(\alpha,\alpha^{-i})]+1)}$. If $x>0$, then

$$\begin{aligned}\overline{(x,\alpha^{-k})}&=\overline{(x-1,\varepsilon)(0,\alpha^{-k})}=\overline{(x-1,\varepsilon)}*\overline{(0,\alpha^{-k})}\\&=\overline{(0,\varepsilon)}^x*\overline{(0,\alpha^{-k})}=\overline{(0,\varepsilon)}^{(x+km+[\sum_{i=1}^k I(\alpha,\alpha^{-i})]+1)}.\end{aligned}$$

We have thus shown that Ker $(h)=\langle\overline{(0,\varepsilon)}\rangle$.

4. A factor system for $G_{(m,\alpha)}$. To completely describe the structure of $G_{(m,\alpha)}$ we have to find a factor system $F: G/\langle\alpha\rangle\times G/\langle\alpha\rangle\to$ Ker (h). In each $\langle\alpha\rangle$-class of G, there is an element γ such that $(0,\gamma)\in S$ is prime relative to (m,α). Fix one such element γ for each $\langle\alpha\rangle$-class. Define a lifting $L: G/\langle\alpha\rangle\to G_{(m,\alpha)}$ by $L(\bar{\gamma})=\overline{(0,\gamma)}$. Then F is defined by the equation $L(\bar{\xi})*L(\bar{\eta})=F(\bar{\xi},\bar{\eta})*L(\bar{\gamma})$ where $\bar{\gamma}=\bar{\xi}*\bar{\eta}$. Note that $\gamma=\alpha^l*\xi*\eta$ for some $l\in Z$, hence $\overline{(0,\xi)}*\overline{(0,\eta)}=F(\bar{\xi},\bar{\eta})*\overline{(0,\alpha^l*\xi*\eta)}$. We want to find the unique prime (w,τ) in S relative to (m,α) such that $F(\bar{\xi},\bar{\eta})=\overline{(w,\tau)}$. Let (v,ρ) be the unique prime in S relative to (m,α) such that $\overline{(v,\rho)}=\overline{(0,\xi)}*\overline{(0,\eta)}$. By Lemma 3.2, (v,ρ) equals

$$\begin{aligned}&(I(\xi,\eta)-jm-\textstyle\sum_{i=1}^j I(\alpha,\alpha^{-i}*\xi*\eta),\alpha^{-j}*\xi*\eta)\\&\quad=(w+I(\tau,\alpha^l*\xi*\eta)-km-\textstyle\sum_{i=1}^k I(\alpha,\alpha^{-i}*\tau*\alpha^l*\xi*\eta),\alpha^{-k}*\tau*\alpha^l*\xi*\eta)\end{aligned}$$

where $j,k\in Z_+^0$ are unique (j is known, but k is not). By equating components and solving for w and τ, we obtain

$$(4.1)\quad\begin{cases}\tau=\alpha^{k-j-l}.\\ w=I(\xi,\eta)+(k-j)m+[\sum_{i=0}^{k-1}I(\alpha,\alpha^i*\xi*\eta)]\\ \qquad -I(\alpha^{k-j-l},\alpha^l*\xi*\eta)-\displaystyle\sum_{i=k-j}^{k-1}I(\alpha,\alpha^i*\xi*\eta).\end{cases}$$

Consequently, our problem is reduced to determining k. Recall that (w,τ) is prime. By using Lemma 3.1 and doing some delicate algebraic manipulations, we obtain the following result.

Theorem 4.2. *A factor system F for the extension $G_{(m,\alpha)}$ of Ker (h) by $G/\langle\alpha\rangle$ is defined by $F(\bar{\xi},\bar{\eta})=\overline{(w,\tau)}$ where τ and w are given by* (4.1). *Furthermore, the non-negative integer k is uniquely determined by $N_{k+1}\leq I(\xi,\eta)<N_k$ where*

$$N_k=I(\alpha^{-l},\alpha^l*\xi*\eta)+(j-k+1)m+[\textstyle\sum_{i=1}^{j+1}I(\alpha,\alpha^{-l-i})-\sum_{i=j-k+2}^{j+1}I(\alpha,\alpha^{-l-i})].$$

References

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[2] Tamura, T.: Basic study of $\mathfrak{N}$-semigroups and their homomorphisms. Semigroup Forum, 8, 21–50 (1974).