58. Identities E-2 and Exponentiality

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1. Introduction. Let S be a semigroup and Z_+ the set of positive integers, and $Z_+^0=Z_+\cup\{0\}$. Define $E(S)$ by

$$E(S)=\{n\in Z_+ : (ab)^n=a^nb^n \quad \text{for all } a, b\in S\}.$$

$E(S)$ is a multiplicative semigroup containing 1. A semigroup is called an E-n semigroup [3] if $n\in E(S)$. If $E(S)=Z_+$, then S is called *exponential.* In some semigroups, E-2 implies exponentiality, for example, this holds for groups, cancellative semigroups or inverse semigroups [4]. More generally, regular E-2 semigroups are exponential [3]. Recently A. Cherubini Spoletini and A. Varisco [1] obtained that a power cancellative E-2 semigroup is exponential and also they have

Proposition 1 ([1]). *Let S be a semigroup. If $n\in E(S)$ then $n+\lambda n(n-1)\in E(S)$ for all $\lambda\in Z_+^0$. Hence if $2\in E(S)$ then $2n\in E(S)$ for all $n\in Z_+$.*

As is well known [2], if S is a group and $E(S)$ contains three consecutive integers then $E(S)=Z_+$. In parallel to this,

Proposition 2 ([5]). *Let S be a semigroup. If 2, n, $n+1$ and $n+2$ are in $E(S)$ then $\{m\in Z_+ : m\geq n\}\subset E(S)$. Therefore, if S is E-2 and E-3, then S is exponential.*

In this paper, Theorem 3 will improve Proposition 2 and the second part of Proposition 1 so that we shall be able to completely describe $E(S)$ when $2\in E(S)$.

2. Results. Theorem 3. *Let S be a semigroup. If $2\in E(S)$, then $m\in E(S)$ for all $m\geq 4$.*

Proof. Since $2\in E(S)$ implies $4\in E(S)$, it is sufficient to verify the following: If $n>2$ and 2, $n\in E(S)$, then $n+1\in E(S)$.

In case n is odd, $n-1$ is even, so let $n-1=2k$. Then

$$\begin{aligned}x^{n+1}y^{n+1}&=x(x^ny^n)y=x(xy)^ny=x^2(yx)^{n-1}y^2\\&=x^2((yx)^k)^2y^2=(x(yx)^k)^2y^2=((xy)^kx)^2y^2\\&=(xy)^{2k}x^2y^2=(xy)^{n-1}(xy)^2=(xy)^{n+1}.\end{aligned}$$

In case n is even, let $n-2=2k$. Then

$$\begin{aligned}x^{n+1}y^{n+1}&=x(x^ny^n)y=x(xy)^ny=x^2(yx)^{n-1}y^2\\&=x^2(yx)^{n-3}(yx)^2y^2=x^2(yx)^{n-3}(yxy)^2\\&=x^2(yx)^{n-2}y^2xy=x^2((yx)^k)^2y^2xy\\&=(x(yx)^k)^2y^2xy=((xy)^kx)^2y^2xy=(xy)^{2k}x^2y^2xy\\&=(xy)^{n-2}(xy)^2(xy)=(xy)^{n+1}.\end{aligned}$$

See in [5] an example of an E-2 semigroup which is not E-3.

Corollary 4. *Let S be an E-2 semigroup. S is exponential if and only if S satisfies the identity $(x^2y)(xy^2)=(xy^2)(x^2y)$.*

A semigroup is called *n-cancellative* if, for $a, b \in S$, $a^n=b^n$ implies $a=b$. If S is n-cancellative for all $n \in Z_+$, then S is called *power-cancellative*. A semigroup is called *n-divisible* if for each $a \in S$ there is $b \in S$ such that $a=b^n$.

For any semigroup S, define semigroups $C(S)$ and $D(S)$ as follows:

$$C(S)=\{n \in Z_+ : S \text{ is } n\text{-cancellative}\}$$

$$D(S)=\{n \in Z_+ : S \text{ is } n\text{-divisible}\}.$$

The following is obtained as a consequence of Remark 1.6 of [1], but the proof here is shorter.

Corollary 5 ([1]). *Assume a semigroup S is either n-cancellative or n-divisible for $n>1$. Then S is exponential if and only if S is E-2.*

Proof. We need only prove sufficiency. Assume S is E-2. Since $n \in C(S)$ implies $n^2 \in C(S)$ [$n \in D(S)$ implies $n^2 \in D(S)$], we can assume $n \neq 3$ without loss of generality.

If S is n-cancellative, by Theorem 3, we see n, $3n \in E(S)$ and then

$$(a^3b^3)^n=a^{3n}b^{3n}=(ab)^{3n}=((ab)^3)^n.$$

By n-cancellation, we have $a^3b^3=(ab)^3$, thus S is E-3. Therefore S is exponential.

If S is n-divisible, $S=\{x^n : x \in S\}$. Then, since $n, 3n \in E(S)$ by Theorem 3,

$$(a^nb^n)^3=((ab)^n)^3=(ab)^{3n}=a^{3n}b^{3n}=(a^n)^3(b^n)^3.$$

Thus S is E-3, hence exponential.

For further study, the following question is raised:

Problem. If S is a semigroup and $3 \in E(S)$, what can we say about $E(S)$?

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