57. Convergence of Approximate Solutions to Quasi-Linear Evolution Equations in Banach Spaces

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1. Introduction. In this paper we consider the Cauchy problem for the quasi-linear equation of evolution

(Q) $du(t)/dt+A(t,u(t))u(t)=0, \quad a.e.\ t\in[0,T], \quad u(0)=a,$

under the following assumptions.

(X) X is a Banach space with norm $\|\cdot\|$. There is another Banach space Y, continuously and densely embedded in X. There is an isomorphism S of Y onto X. The norm $\|\cdot\|_Y$ in Y is chosen so that S becomes an isometry.

(I) For each $t\in[0,T_0]$ and $y\in W_R$, $-A(t,y)$ is the infinitesimal generator of a (C_0) semigroup $\{\exp[-sA(t,y)]\}_{s\geq 0}$ on X such that $\|\exp[-sA(t,y)]\|\leqq e^{\alpha s}$, where T_0, α and R are positive constants and $W_R=\{y\in Y: \|y\|_Y\leqq R\}$.

(II) For each $t\in[0,T_0]$ and $y\in W_R$, there is a bounded linear operator $B(t,y)$ on X into itself such that $SA(t,y)S^{-1}=A(t,y)+B(t,y)$, $\|B(t,y)\|\leqq\lambda$, where λ is a positive number independent of $t\in[0,T_0]$ and $y\in W_R$.

(III) For each $t\in[0,T_0]$ and $y\in W_R$, we have $D(A(t,y))\supset Y$. The restriction of $A(t,y)$ to Y (which is a bounded linear operator on Y into X by the closed graph theorem) satisfies the following:

$$\|A(t,y)-A(t,z)\|_{Y,X}\leqq\mu\|y-z\|, \qquad t\in[0,T_0],\ y,z\in W_R,$$

where μ is a positive constant and $\|\cdot\|_{Y,X}$ is the operator norm in the Banach space of all bounded linear operators on Y into X.

(IV) For each $y\in W_R$ and $x\in Y$, $t\to A(t,y)x$ is continuous in X.

Assumptions (I) and (II) imply that $\exp[-sA(t,y)](Y)\subset Y$ and the restriction of $\exp[-sA(t,y)]$ to Y is a (C_0) semigroup on Y such that $\|\exp[-sA(t,y)]\|_Y\leqq e^{\gamma s}$, where γ is a positive constant. See [1]. Assumption (IV) is somewhat weaker than the corresponding assumption of [1]. It is assumed in [1] that $t\to A(t,y)$ is continuous in $\|\cdot\|_{Y,X}$-norm.

Using the perturbation theory for the linear equation of evolution, Kato [1] studied in detail the Cauchy problem for the quasi-linear equation of evolution. The purpose of this note is to show another approach to (Q). In § 2, we construct approximate solutions to (Q) for some $T\in(0, T_0)$, and then we show the convergence of them. In § 3, we prove that the limit function obtained in § 2 is a unique solution to (Q) if X is reflexive.

2. Construction and convergence of approximate solutions. Let $r\in(0, R)$ and let $a\in W_r$. We choose a positive number T such that $T<\min\{T_0, \beta^{-1}\log R/r\}$, where $\beta=\max\{\alpha, \gamma\}$. Let $P=\{t_k\}$ be a strictly increasing sequence in $[0, T]$. We define $\{x_k\}$ as follows:

$$x_0=a, \quad x_{k+1}=\exp[-(t_{k+1}-t_k)A(t_k, x_k)]x_k, \quad k=0, 1, \cdots.$$

Let $t_\infty=\lim_{k\to\infty} t_k$. For each $t\in[t_0, t_\infty)$ and $s\in[t_0, t]$, we define linear operator $U(t, s)$ as follows:

$$U(t, s)=\exp[-(t-t_k)A(t_k, x_k)]\cdots\exp[-(t_{j+1}-s)A(t_j, x_j)],$$

if $t\in[t_k, t_{k+1}]$ and $s\in[t_j, t_{j+1}]$. By the method used in [2], we obtain the following

Lemma 2.1. *$\{x_k\}$ converges in Y-norm as $k\to\infty$.*

Proof. Let $t_k>t_j>t_i>t_0$ and let $y\in Y$. Then we have

$$\begin{aligned}\|x_k-x_j\|_Y=\|Sx_k-Sx_j\|&\leqq\|Sx_k-SU(t_k, t_i)S^{-1}y\|\\&+\|SU(t_k, t_i)S^{-1}y-U(t_k, t_i)y\|+\|U(t_k, t_i)y-U(t_j, t_i)y\|\\&+\|U(t_j, t_i)y-SU(t_j, t_i)S^{-1}y\|+\|SU(t_j, t_i)S^{-1}y-Sx_j\|\\&=I_1+I_2+I_3+I_4+I_5.\end{aligned}$$

Now, since $x_k=U(t_k, t_i)x_i$ and $x_j=U(t_j, t_i)x_i$, we have $I_1=\|U(t_k, t_i)x_i-U(t_k, t_i)S^{-1}y\|_Y\leqq e^{\beta T}\|Sx_i-y\|$ and $I_5\leqq e^{\beta T}\|Sx_i-y\|$. I_3 is bounded above by

$$\begin{aligned}&\sum_{p=j}^{k-1}\|U(t_{p+1}, t_i)y-U(t_p, t_i)y\|\\&=\sum_{p=j}^{k-1}\|[\exp[-(t_{p+1}-t_p)A(t_p, x_p)]-1]U(t_p, t_i)y\|\\&=\sum_{p=j}^{k-1}\left\|\int_0^{t_{p+1}-t_p}\exp[-rA(t_p, x_p)]A(t_p, x_p)U(t_p, t_i)y\,dr\right\|,\end{aligned}$$

and since $\sup_{t,y}\|A(t, y)\|_{Y,X}<\infty$, we have $I_3\leqq C(t_k-t_j)\|y\|_Y$. Next, we put $A[r]=A(t_p, x_p)$ and $B[r]=B(t_p, x_p)$ if $r\in[t_p, t_{p+1}]$, $p=0, 1, \cdots$. Then we have

$$\begin{aligned}I_2&=\left\|\int_{t_i}^{t_k}(d/dr)[SU(t_k, r)S^{-1}U(r, t_i)y]dr\right\|\\&\leqq\int_{t_i}^{t_k}\|SU(t_k, r)(A[r]S^{-1}-S^{-1}A[r])U(r, t_i)y\|\,dr\\&=\int_{t_i}^{t_k}\|SU(t_k, r)S^{-1}B[r]U(r, t_i)y\|\,dr\leqq C(t_k-t_i)\|y\|,\end{aligned}$$

and in the same way, we have $I_4\leqq C(t_j-t_i)\|y\|$. Thus we have $\overline{\lim}_{j,k\to\infty}\|x_k-x_j\|_Y\leqq C\|Sx_i-y\|+C(t_\infty-t_i)\|y\|$, for every i and $y\in Y$. This implies that $\overline{\lim}_{j,k\to\infty}\|x_k-x_j\|_Y=0$. Q.E.D.

Lemma 2.2. *For each $\varepsilon>0$ and $a\in W_r$, there is a partition $P(\varepsilon, a): 0=t_0<t_1<\cdots<t_{N_\varepsilon}=T$, of $[0, T]$ such that*

(i) $t_{k+1}-t_k\leqq\varepsilon$, $k=0, 1, \cdots, N$,

(ii) $\|[A(t, x_k)-A(t_k, x_k)]\exp[-(t'-t_k)A(t_k, x_k)]x_k\|\leqq\varepsilon$, *for* t, t'

$\in [t_k, t_{k+1}]$, $k=0, \cdots, N_\varepsilon$,
where $x_0=a$ *and* $x_{k+1}=\exp[-(t_{k+1}-t_k)A(t_k, x_k)]x_k$, $k=0, 1, \cdots, N_\varepsilon$.

Proof. Inductively, we define $\{t_k\}$ and $\{x_k\}$ in the following manner: Suppose that t_j and x_j, $j=0, \cdots, k$ are constructed. Then if $t_k<T$, let t_{k+1} be the largest number satisfying (i), (ii) and $t_{k+1}\leqq T$, and let $x_{k+1}=\exp[-(t_{k+1}-t_k)A(t_k, x_k)]x_k$. Note that $t_{k+1}>t_k$. We shall prove that there is an N such that $t_N=T$. Assume, for the contrary, that $t_k<T$ for all $k=0, 1, \cdots$. Let $t_\infty=\lim_{k\to\infty} t_k$ and let $w\in W_R$ be the limit point of $\{x_k\}$. Then, for every $t, t'\in[t_k, t_\infty]$, we have

$$\begin{aligned} M_k &\equiv \|[A(t, x_k)-A(t_k, x_k)]\exp[-(t'-t_k)A(t_k, x_k)]x_k\| \\ &\leqq \|[A(t, x_k)-A(t, w)]\exp[-(t'-t_k)A(t_k, x_k)]x_k\| \\ &\quad +\|[A(t, w)-A(t_k, w)]\exp[-(t'-t_k)A(t_k, x_k)]x_k\| \\ &\quad +\|[A(t_k, w)-A(t_k, x_k)]\exp[-(t'-t_k)A(t_k, x_k)]x_k\| \\ &\leqq 2\mu R\|x_k-w\|+\|[A(t, w)-A(t_k, w)]\exp[-(t'-t_k)A(t_k, x_k)]x_k\|, \end{aligned}$$

and since $\lim_{k\to\infty}\|\exp[-(t'-t_k)A(t_k, x_k)]x_k-w\|_Y=0$, we have $\lim_{k\to\infty} M_k=0$. Therefore, for every $\varepsilon>0$, there is a k such that $t_\infty-t_k<\varepsilon$ and $M_k<\varepsilon$. On the other hand, since t_{k+1} is the largest number satisfying (i) and (ii), we have $t_{k+1}>t_\infty$. This contradicts $t_{k+1}\leqq t_\infty$. Q.E.D.

Let $\varepsilon=1/n$ and $a\in W_r$, and let $P(1/n, a): 0=t_0^n<t_1^n<\cdots<t_{N_n}^n=T$ be the partition of $[0, T]$ constructed by Lemma 2.2. We put $x_0^n=a$, $x_{k+1}^n=\exp[-(t_{k+1}^n-t_k^n)A(t_k^n, x_k^n)]x_k^n$ and $u_n(t)=\exp[-(t-t_k^n)A(t_k^n, x_k^n)]x_k^n$ if $t\in[t_k^n, t_{k+1}^n]$, $k=0, 1, \cdots, N_n$. Then we have

Proposition 2.3. $\{u_n(t)\}$ *converges in* X *as* $n\to\infty$, *uniformly in* $t\in[0, T]$.

Proof. Let $t\in(t_k^n, t_{k+1}^n)\cap(t_j^m, t_{j+1}^m)$. Suppose that $t_k^n\geqq t_j^m$. Then we have

$$\begin{aligned} &(d/dt)\|u_n(t)-u_m(t)\|^2 \\ &\quad = -2(A(t_k^n, x_k^n)(u_n(t)-u_m(t)), f)-2((A(t_k^n, x_k^n)-A(t_k^n, x_j^m))u_m(t), f) \\ &\qquad -2((A(t_k^n, x_j^m)-A(t_j^m, x_j^m))u_m(t), f) \\ &\quad \leqq 2\beta\|u_n(t)-u_m(t)\|^2+2\mu R\|x_k^n-x_j^m\|\,\|u_n(t)-u_m(t)\| \\ &\qquad +(2/m)\|u_n(t)-u_m(t)\|, \end{aligned}$$

where $f\in F(u_n(t)-u_m(t))$, $F: X\to X^*$ is the duality mapping (multi-valued). The second term of the right hand side of the above inequality is bounded above by

$$\begin{aligned} &2\mu R\,[\|x_k^n-u_n(t)\|+\|u_n(t)-u_m(t)\|+\|u_m(t)-x_j^m\|]\,\|u_n(t)-u_m(t)\| \\ &\qquad \leqq C(1/n+1/m)\|u_n(t)-u_m(t)\|+2\mu R\|u_n(t)-u_m(t)\|^2. \end{aligned}$$

Therefore, we have $\|u_n(t)-u_m(t)\|\leqq C(1/n+1/m)$. Q.E.D.

3. Existence of a local solution. In this section, we assume that X is reflexive. Then, since S is an isomorphism, Y is also reflexive. Therefore, W_R is closed in X. See [1, Lemma 7.3]. Now, we shall prove the following

Theorem 3.1. *Let* $u(t)$ *be the limit function of* $\{u_n(t)\}$ *obtained* *by Proposition* 2.3. *Then $u(t)$ is a unique solution to* (Q) *if X is reflexive.*

Strictly speaking, an X-valued function $u(t)$ on $[0, T]$ is called *a solution to* (Q) if $u(t)$ is strongly absolutely continuous, $u(t)$ is strongly differentiable at almost every $t \in [0, T]$, $u(t) \in W_R$ for almost every $t \in [0, T]$ and $u(t)$ satisfies (Q).

Proof of Theorem 3.1. We first note that $u(t) \in W_R$, because $u_n(t) \in W_R$ and W_R is closed in X. Furthermore, since $(d/dt)u_n(t) = -A(t_k^n, x_k^n)u_n(t)$, $t \in [t_k^n, t_{k+1}^n]$ and $\sup_{t,y} \|A(t, y)\|_{Y,X} < \infty$, $\|(d/dt)u_n(t)\|$ is uniformly bounded in $t \in [0, T]$ and n. Therefore, $u(t)$ is Lipschitz continuous in $t \in [0, T]$ and $u(t)$ is strongly differentiable at almost every $t \in [0, T]$. The uniqueness of the solution can be proved as usual. Thus the following lemma leads to the conclusion of Theorem 3.1.

Lemma 3.2. *If $u(t)$ is differentiable at $s \in (0, T]$, then $u'(s) = -A(s, u(s))u(s)$.*

Proof. For every $y \in Y$, we have

$$(d/dt)\|u_n(t) - y\|^2 = -2(A_n[t]u_n(t), f) \leqq 2\beta\|u_n(t) - y\|^2 + 2\langle -A_n[t]y, u_n(t) - y\rangle_s,$$

where $f \in F(u_n(t) - y)$, $\langle p, q\rangle_s = \sup\{(p, f) : f \in F(q)\}$, $p, q \in X$ and $A_n[t] = A(t_k^n, x_k^n)$ if $t \in [t_k^n, t_{k+1}^n)$. Integrating each side of this inequality from s to t and then passing to the limit as $n \to \infty$, we have

$$\|u(t) - y\|^2 - \|u(s) - y\|^2 \leqq 2\beta \int_s^t \|u(r) - y\|^2 \, dr + 2\int_s^t \langle -A(r, u(r))y, u(r) - y\rangle_s \, dr.$$

Therefore, since $(u(t) - u(s), f) \leqq (1/2)(\|u(t) - y\|^2 - \|u(s) - y\|^2)$ for every $f \in F(u(s) - y)$, we have

$$(u'(s), g) \leqq \beta\|u(s) - y\|^2 + (-A(s, u(s))y, g),$$

for some $g \in F(u(s) - y)$. See [3, Lemma 1]. On the other hand, since $u(s-h) = u(s) - hu'(s) + o(h)$ as $h \downarrow 0$ and since $A(s, u(s)) + \beta$ is m-accretive in Y, there is a $y_h \in Y$ such that $(1 + hA(s, u(s)))y_h = u(s) - hu'(s) + o(h)$. Thus we have

$$(h^{-1}(u(s) - y_h) - A(s, u(s))y_h + o(1), g) \leqq \beta\|u(s) - y_h\|^2 + (-A(s, u(s))y_h, g),$$

for some $g \in F(u(s) - y_h)$. This implies that $\|u(s) - y_h\| = o(h)$ as $h \downarrow 0$. Thus we have $y_h \to u(s)$ and $A(s, u(s))y_h \to -u'(s)$ as $h \downarrow 0$, and since $A(s, u(s))$ is closed, we have $-u'(s) = A(s, u(s))u(s)$. Q.E.D.

References

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