55. *Parametrix for a Degenerate Parabolic Equation*

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§ 1. **Introduction.** The purpose of this note is to construct a parametrix of a Cauchy problem 1) for a parabolic equation with a degenerate principal symbol:

$$1)\quad \begin{cases} \left(\dfrac{\partial}{\partial t}+p(x,D)\right)u(x,t)=f(x,t) & \text{on } \boldsymbol{R}^n\times[0,T], \\ u(x,0)=g(x) & \text{on } \boldsymbol{R}^n, \end{cases}$$

$p(x,D)$ being a pseudo-differential operator whose symbol $p(x,\xi)$, independent of t, is in $S^m_{1,0}=L^m_{1/2}$ and has an asymptotic behavior 2);

2) $p(x,\xi)\sim p_m(x,\xi)+p_{m-1}(x,\xi)+\cdots$ as $|\xi|\to\infty$,

$p_\nu(x,\xi)$ being positively homogeneous in ξ of order ν for $|\xi|\geqq 1$ and the principal symbol p_m being real non-negative ($m>1$).

Melin's result [4] proves the existence of fundamental solution E for 1) under some condition for subprincipal symbols using functional methods. Our method is direct. Under the same condition a complex phase function is given at first in a simple function of a principal symbol and a subprincipal symbol, and amplitudes follow inductively. Consequently a parametrix represented by pseudo-differential operators in $S^0_{1/2\,1/2}=L^0_0$ is gotten. The parametrix implies the existence of fundamental solution and also Melin's result as a corollary.

§ 2. **Notations.** Here we employ the Weyl symbol for pseudo-differential operators, that is, a symbol $a(x,\xi)$ defines an operator $a(x,D)$ by 3):

$$3)\quad a(x,D)u(x)=(2\pi)^{-n}\iint e^{i(x-y)\xi}a\left(\frac{x+y}{2},\xi\right)u(y)dyd\xi \text{ for } u\in C_0^\infty.$$

Hence p_{m-1} is the subprincipal symbol in usual sence. $\nabla^k a$ stands for a section of $T^{*k}(T^*\boldsymbol{R}^n)$, k-th symmetric tensor of $T^*(T^*\boldsymbol{R}^n)$, defined by 4) with respect to the canonical coordinate of $T^*\boldsymbol{R}^n$:

$$4)\quad \sum_{|\alpha+\beta|=k} c^k_{\alpha\beta}a^{(\alpha)}_{(\beta)}(d\xi)^\alpha(dx)^\beta;$$

$$c^k_{\alpha\beta}=k!/\alpha!\,\beta! \quad\text{and}\quad a^{(\alpha)}_{(\beta)}=\langle\xi\rangle^{(|\alpha|-|\beta|)/2}\partial^\alpha_\xi\partial^\beta_x a(x,\xi).$$

σ^1 is the canonical two form $d\xi\wedge dx$ on $T^*\boldsymbol{R}^n$. σ^k is its extension onto $T^k(T^*\boldsymbol{R}^n)\times T^k(T^*\boldsymbol{R}^n)$. J_k is the identification map of $T^{*k}(T^*\boldsymbol{R}^n)$ to $T^k(T^*\boldsymbol{R}^n)$ defined by $\sigma^k(u,J_kf)=\langle u,f\rangle$. A bilinear form $\langle J_kf,g\rangle$ on $T^{*k}(T^*\boldsymbol{R}^n)$ is denoted by $\sigma_k(f,g)$. A linear map defined by $\nabla^k a$ from

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$T^j(T^*\boldsymbol{R}^n)$ to $T^{*k-j}(T^*\boldsymbol{R}^n)$ is denoted by the same notation $\nabla^k a$.

For the principal symbol p_m, the Hamilton vector field $h=J_1\nabla p_m$ and the Hamilton matrix $-iA=J_1\nabla^2 p_m$ are well defined. $-iA$ is a linear map on $T(T^*\boldsymbol{R}^n)$. $\tilde{T}r\,A$ stands for the positive trace, namely, the sum of real positive eigenvalues of A.

Proposition 1. *If $a_i \in S^{m_i}_{\rho_i\delta_i}$, $i=1,2$ and $\rho_i>\delta_{3-i}$, then the symbol $a_1\circ a_2$ of the product operator $a_1(x,D)a_2(x,D)$ has the asymptotic expansion* 5) *where $\sigma_0(a_1,a_2)=a_1a_2$:*

5) $a_1\circ a_2 \equiv \sum_{k=0}^{\infty}(2i)^{-k}(k!)^{-1}\sigma_k(\nabla^k a_1, \nabla^k a_2) \bmod S^{-\infty}$.

Remark. $\sigma_k(\nabla^k a_1, \nabla^k a_2)$ may be denoted by $\sigma(\nabla^k a_1, \nabla^k a_2)$ or $\sigma_k(a_1, a_2)$.

§3. Parametrix. We assume 6).

6) $p_m \geqq 0$ on $T^*\boldsymbol{R}^n$ and $2\,\mathrm{Re}\,p_{m-1}+\tilde{T}rA \geqq c\,|\xi|^{m-1}$ on the characteristic Σ of p_m for a positive constant c.

The parametrix is formed as the sum of two parts e and e' having asymptotic expansions 7):

7) $E_\rho=e+e'$, $e\sim\sum_{i=0}^{\infty}e_i$, $e'\sim\sum_{i=0}^{\infty}e'_i$, $e_i=f_i\exp\varphi$ and $e'_i=f'_i\exp\varphi$.

Here φ is given in 8) and f_0 and f'_0 are given in 12).

8) $\varphi=\psi_1\varphi_1+(1-\psi_1)\varphi_2$

9) $\varphi_1=-p_mt-p_{m-1}t-\langle\nabla p_m, F(At/2)J_1\nabla p_m\rangle t^2$
$\quad -2^{-1}\,\mathrm{Tr}\,(\log[\cosh(At/2)])$

10) $F(\lambda)=(4i\lambda)^{-1}(1-\lambda^{-1}\tanh\lambda)$

11) $\varphi_2=-p_mt-\langle\xi\rangle^{m-1-\gamma}t$

12) $f_0=\psi_2$ and $f'_0=1-\psi_2$

13) $\psi_k=\psi^1_k\psi^2_k$, $\psi^1_k=\psi(k^{-1}p_m\langle\xi\rangle^{1-m-\varepsilon})$, $\psi^2_k=\psi(k^{-1}t\langle\xi\rangle^{m-1-\delta})$, and $\psi\in C^\infty[0,\infty)$ such that $\psi=1$ if $s\leqq 1$, $\psi'<0$ if $1<s<2$, $\psi=0$ if $s\geqq 2$ and $|\psi^{(n)}|\leqq C_{n\tau}(1-\psi)^\tau$ if $0<\tau<1$.

14) $0\leqq 12\gamma<12\delta<1-3\varepsilon<1$.

Theorem 1. *Under the assumption* 6), *if φ, f_0 and f'_0 are defined by* 8)–13) *and if γ,δ and ε satisfy* 14), *then there exist $\{f_j\}$ and $\{f'_j\}$ such that* $\mathrm{supp}\,f_j\subset\mathrm{supp}\,\psi_2$ *and* $\mathrm{supp}\,f'_j\subset\mathrm{supp}\,(1-\psi_2)$ *and that* $E_p=e(t,x,D)+e'(t,x,D)$ *defined by* 7) *is a parametrix of the problem* 1) *for some $T>0$, that is, E_p satisfies* 15). *Each e_j and e'_j belongs to $S^{-\varepsilon j/2}_{1/2\,1/2}$.*

15) $(\partial_t-p)E_p\equiv E_p(\partial_t-p)\equiv 0$ on $[0,T]$ mod $S^{-\infty}$ and $E_p|_{t=0}=I$.

Remark. The condition 14) guarantees that $F(At/2)$ and $\cosh(At/2)$ are well defined and that $\exp\varphi$ belongs to $S^0_{1/2\,1/2}$.

The following two lemmas are important for the proof of Theorem 1. Equations in these lemmas are approximations of those corresponding to the Hamilton-Jacobi equation and the transport equation in real cases.

Notations. Classes of symbols $N(j,k,l)$ are defined by steps through 16) and 17). (They are considered only on supp ψ_2.)

16) f belongs to $N(0,0,l)$ if and only if f is a C^∞-function in (t,x,ξ) such that for all integers $i,j\geqq 0$ and for some constants c_{ij} and d_{ij}

$$|\partial_t^i \nabla^j f|\leqq c_{ij}(1+t\langle\xi\rangle^{m-1})^{d_{ij}}\langle\xi\rangle^{(l-j\varepsilon+2i(m-1))/2}$$

17) $N(j,k,l)$ (j and k are integers such that $j\geqq\max(k,0)$) consist of C^∞-functions f such that $(t\langle\xi\rangle^{m-1})^{k-j}f$ is a polynomial of homogeneous order k in $\zeta=tJ_1\nabla p_m$ with coefficients in $N(0,0,l)$ if $k\geqq 0$ and that $(t\langle\xi\rangle^{m-1})^{-j}f$ belongs to $N(0,0,l+k\varepsilon)$ if $k<0$. ($N(j,k,l)$ are the empty set if $j<\max(k,0)$.)

Lemma 1. *Let φ_1 be defined by* 9). *Then, φ_1 satisfies* 18) *with* $g\in\sum_{i=0}^{3}N(i,i,2m-3)$.

18) $$\frac{d}{dt}(\exp\varphi_1)+\sum_{\nu=0}^{2}(2i)^{-\nu}(\nu!)^{-1}\sigma_\nu(p_m,\exp\varphi_1)+p_{m-1}\exp\varphi_1=g\exp\varphi_1.$$

Lemma 2. *For $g\in N(j,k,l)$ there exists $f\in N(j+1,k,l+2-2m)$ which satisfies* 19) *and* 20):

19) $f|_{t=0}=0$.

20) $$\frac{d}{dt}f+\sum_{\nu=1}^{2}(2i)^{-\nu}(\nu!)^{-1}\{\sigma_\nu(p_m,f\exp\varphi_1)-\sigma_\nu(p_m,\exp\varphi_1)f\}\exp(-\varphi_1)$$
$$\equiv g \mod N(j+1,k-2,l)+N(j+1,k+1,-\varepsilon+l).$$

§4. Fundamental solution. Once a parametrix has been constructed, the Green operator E is easily obtained by solving a Volterra's integral equation 21) of pseudo-differential operators. It is shown by Proposition 2 that E is represented as a pseudo-differential operator.

21) $$E(t)+\int_0^t E(t-s)G_N(s)ds=E_N(t),$$

where

$$E_N(t)=\sum_{i=0}^{N}(e_i+e_i') \quad\text{and}\quad G_N(t)=\left(\frac{d}{dt}+p\right)E_N(t).$$

Proposition 2. *Let p_j ($j=1,\cdots,\nu$) be in $L_0^{m_j}$. Then $p=p_1\circ\cdots\circ p_\nu$ is in $L_0^{m_0}$ $\left(m_0=\sum_{j=1}^{\nu}m_j\right)$ and satisfies* 22) *for all integer $l\geqq 0$ and for some integer l_0 and constant C_l which are dependent on l but independent of ν.*

22) $$|p|_l^{(m_0)}\leqq(C_l)^\nu\prod_{j=1}^{\nu}|p_j|_{l_0}^{(m_j)}$$

where

$$|p|_l^{(m)}=\max_{k\leqq l}\left\{\sup_{(x,\xi)\in R^{2n}}|\nabla^l p(x,\xi)|\langle\xi\rangle^{-m}\right\}.$$

(Refer to C. Iwasaki [3].)

Lemma 3. $G_N(t)\in L_0^{m-1-(N+1)\varepsilon/2}$.

Theorem 2. *There exists a pseudo-differential operator $H_N(t)$*

$\in L_0^{m-1-(N+1)\varepsilon/2}$ *such that* $E(t)=E_N(t)+\int_0^t H_N(t-s)E_N(s)ds$ *is the unique solution of* 23), *that is,* $E(t)\delta_x$ *is the fundamental solution of the Cauchy problem* 21) *in* $\mathcal{S}'(\boldsymbol{R}^n)$.

Let $p=p_m+p_{m-1}$ be real. $(p+\lambda)\int_0^T e^{-\lambda s}E(s)ds=\int_0^T e^{-\lambda s}E(s)ds(p+\lambda)$ $=I-e^{-\lambda T}E(T)$. This means that $p+\lambda$ on $\mathcal{S}$ has the unique positive selfadjoint extension on $L^2(\boldsymbol{R}^n)$ for a sufficiently large constant λ.

Corollary (A. Melin [4]). *There exists a real* λ *such that* $\mathrm{Re}\,((p+\lambda)u, u)\geqq 0$ *for* $u\in\mathcal{S}$.

§ 5. **Remarks.** (1) We consider a more restrictive case. p_m vanishes exactly to second order on Σ, that is, $p_m(X)\geqq c(X)d(X,\Sigma)^2$, $(X=(x,\xi))$, for a continuous function $c(X)>0$ $(\xi\neq 0)$ where $d(X,\Sigma)$ is the distance of X to Σ in $\boldsymbol{R}^n\times\boldsymbol{R}\times S^{n-1}$. (Refer to L. Hörmander [2].) In this case Σ is necessarily a C^∞-submanifold of $T^*\boldsymbol{R}^n\backslash\{0\}$. Therefore $d(X,\Sigma)$ is a C^∞-function at a neighborhood of Σ and there exists a C^∞-mapping $a(X)$ valued in Σ such that $d(X,a(X))=d(X,\Sigma)$. Let χ be a mapping such that $\chi(a-X)=(\langle\eta\rangle^{1/2}(y-x),\ \langle\eta\rangle^{-1/2}(\eta-\xi))\in T_a(T^*\boldsymbol{R}^n)$ where $(y,\eta)=a(X)$. Then, we can replace the phase function φ_1 at a neighborhood of Σ with φ_3 defined in 23). If we add a condition that $4\varepsilon\leqq 1$, Theorem 1 is valid for the same ε on any compact set of $\boldsymbol{R}^n$.

23) $$\varphi_3=-p_{m-1}(a)t+i\sigma^1(\chi(a-X),\tanh(A(a)t/2)\chi(a-X)) -2^{-1}\,\mathrm{Tr}\,(\log[\cosh(A(a)t/2)]).$$

(φ_3 of the simplest p_m is found in C. Hoel [1].)

(2) Considering the problem 1) on a compact C^∞-manifold, A. Menikoff and J. Sjöstrand [5] computed the rate of $\mathrm{Tr}\,E$ in t as t tended zero adding still more the condition that Σ is simplectic. Using the result of above Remark (1) we can get the same rate without this condition. $\mathrm{Tr}\,E=(c_1+o(1))t^{-n/m}$, $(c_2+o(1))t^{-n/m}\log t$ or $(c_3+o(1))t^{-(n-d)/(m-1)}$ depending on $d=\frac{1}{2}\,\mathrm{codim}\,\Sigma$ such that $md-n\gtreqless 0$.

References

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