54. *On Unimodal Linear Transformations and Chaos*

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§ 0. **Introduction.** Recently, there appeared many works which investigate how the orbit $\{f_\mu^n(x)\,;\, n\geq 0\}$ starting from an initial point $x\in[0,1]$ behaves asymptotically for a family of continuous maps f_μ from the interval $[0,1]$ into itself with a parameter μ. [1]–[6].

In the present paper we treat the unimodal linear transformations as a simple case of such maps f_μ. In general, we call a continuous map f from $[0,1]$ into itself a unimodal linear transformation if f takes the extremum at c and f is linear on each intervals $[0,c]$ and $[c,1]$, for some $c\in(0,1)$. But we only treat the maps defined by

Definition 0.1. *Let $a>0$, $b>1$ and $a+b-ab\geqq 0$. Let a unimodal linear transformation f_μ with parameter $\mu=(a,b)$ be*

$$(1)\qquad f_\mu(x)=\begin{cases} ax+\dfrac{a+b-ab}{b} & \textit{for } 0\leqq x\leqq 1-\dfrac{1}{b} \\ -b(x-1) & \textit{for } 1-\dfrac{1}{b}\leqq x\leqq 1. \end{cases}$$

It is not difficult to see that the general unimodal linear transformations are essentially reduced to f_μ of the Definition 0.1, with some trivial exceptions.

In the present paper we state the results only. The proofs of these results will be given in forthcoming papers "On unimodal linear transformations and chaos. I, II" which will appear in Tokyo Journal. We will treat the case $a=b$ in I and the general case in II in detail.

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§ 1. **Some notations and definitions.** For f_μ defined by (1), let a pair of intervals $\{I_0, I_1\}$, which we will call the fundamental partition of f_μ, be as follows:

Let $I_0=\left[0,1-\dfrac{1}{b}\right]$ and $I_1=\left(1-\dfrac{1}{b},1\right]$ in the case when $f_\mu^n(0)=0$, $f_\mu^i(0)\neq 0$ for $1\leqq i\leqq n-1$ for some natural number n, and the number k defined by

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(2) $$k=\#\left\{i\,;\,1\leqq i\leqq n-3,\,1-\frac{1}{b}<f_\mu^i(0)\right\}$$

is odd. Let $I_0=\left[0,1-\frac{1}{b}\right)$ and $I_1=\left[1-\frac{1}{b},1\right]$ otherwise.

Using this fundamental partition, we can realize f_μ as a symbolic dynamical system as follows. Denote by Ω the cartesian product space $\{0,1\}^{N^*}$, where $N^*=N\cup\{0\}$, with the usual topology, and denote by σ the shift operator on Ω. Define a map π_μ from $[0,1]$ into Ω by

(3) $$\pi_\mu(x)(n)=i \quad \text{if } f_\mu^n(x)\in I_i \quad (i=0 \text{ or } 1)$$

for any $x\in[0,1]$ and $n\geqq 0$, where $\pi_\mu(x)(n)$ denotes the n-th coordinate of $\pi_\mu(x)$, and call π_μ the realization of f_μ on Ω. Let $Y_\mu=\pi_\mu[0,1]$ and X_μ be the closure of Y_μ in Ω. Let us introduce the order relation in Ω. Let $S(n,\omega)$ be

(4) $$\begin{cases} S(n,\omega)=(-1)^{\sum_{i=1}^{n-1}\omega(i)} & (n\geqq 1)\\ S(0,\omega)=1. \end{cases}$$

For $\omega,\omega'\in\Omega$, $\omega<\omega'$ means that, for some n,

(5) $$\begin{cases} \omega(i)=\omega'(i) & \text{for } 1\leqq i\leqq n-1\\ \omega(n)<\omega'(n) & (\omega(n)>\omega'(n)) \text{ if } S(n,\omega)=1\ (=-1, \text{ respectively}). \end{cases}$$

Theorem 1.1. *Using this order relation, we can characterize X_μ by the image of 0 under the realization π_μ as follows:*

(6) $$X_\mu=\{\omega\in\Omega\,;\,\pi_\mu(0)\leqq\sigma^n\omega \text{ for any } n\geqq 0\}.$$

π_μ is not one-to-one in general, that is to say, the fundamental partition is not a generator in general.

To state the results, we divide the domain of parameter

(7) $$D=\{(a,b)\,;\,a>0,\ b>1,\ a+b-ab\geqq 0\}$$

as follows:

(8) $$\begin{aligned}
&D_0=\{(a,b)\in D\,;\,ab\leqq 1\}\\
&D_1=\left\{(a,b)\in D\,;\,ab>1,\,\frac{a+b-ab}{b}\geqq\frac{b}{b+1}\right\}\\
&D_2=\left\{(a,b)\in D\,;\,ab>1,\,\frac{b}{b+1}>\frac{a+b-ab}{b}\geqq\frac{b-1}{b}\right\}\\
&D^{(k)}=\{(a,b)\in D\,;\,ab>1,\ a<1,\ 1+a^{-1}+\cdots+a^{-(k-1)}<b\\
&\qquad<1+a^{-1}+\cdots+a^{-k}\},\qquad k\geqq 2\\
&D_3^{(k)}=\{(a,b)\in D^{(k)}\,;\,b\leqq a^{-k}\},\qquad k\geqq 2\\
&D_4^{(k)}=\{(a,b)\in D^{(k)}\,;\,a^{-k}<b,\ a+b\geqq a^kb^2\},\qquad k\geqq 2\\
&D_5^{(k)}=D^{(k)}-(D_3^{(k)}\cup D_4^{(k)}),\qquad k\geqq 2\\
&D_6=\left\{(a,b)\in D\,;\,a>1,\ b>1,\,\frac{b}{b+1}>\frac{a+b-ab}{b}\right\}.
\end{aligned}$$

§ 2. Main results. Now we can state main results.

First of all, note that the fundamental partition is not a generator in the cases D_0 and $D_3^{(k)}$. In these cases we have

Theorem 2.1. *If $\mu=(a, b)\in D_0$, then every point of* $[0,1]$ *is asymptotically periodic. More precisely,*

(i) *If $ab<1$, then there exists a periodic orbit with period* 2 *and every orbit approaches this periodic orbit.*

(ii) *If $ab=1$, then there exist intervals A_0 and A_1 such that* (a) *$f_\mu A_0=A_1$ and $f_\mu A_1=A_0$* (b) *$A_0\cup A_1$ consists of one periodic orbit with period* 2 *and periodic orbits with period* 4. (c) *For any $x\in[0,1]-(A_0\cup A_1)$, $f_\mu^n(x)\in A_0\cup A_1$ for some n. And the topological entropy of f_μ is equal to* 0.

Theorem 2.2 (the case of "window"). *If $\mu=(a, b)\in D_3^{(k)}$, then*

(i) *there exists an interval A_0 which satisfies $f_\mu^{k+1}A_0\subset A_0$ and, for any $x\in[0,1]-A_0$, $f_\mu^n(x)\in A_0$ for some n.*

(ii) *In particular, in the case when $1+a^{-1}+\cdots+a^{-(k-1)}<b\lneqq a^{-k}$, there exists a periodic orbit with period $k+1$ and almost all (with respect to the Lebesgue measure) orbits approach this periodic orbit.*

This is the case of so called "window". Note that f_μ has a periodic point of period 3, and consequently, periodic points of any period in this case. And except for the case when μ is on the boundary of $D_3^{(k)}$, f_μ has no invariant measure which is absolutely continuous with respect to the Lebesgue measure.

Theorem 2.3 (topological entropy in the case of "window"). *If $\mu=(a, b)\in D_3^{(k)}$, then the topological entropy of f_μ is equal to $\log\gamma_k$, where γ_k is the maximal root of the equation $\gamma^k-\gamma^{k-1}-\cdots-\gamma-1=0$.*

In the case of $D-\left(D_0\cup\bigcup_{k=2}^{\infty} D_3^{(k)}\right)$, the fundamental partition is a generator, so we obtain the so-called f_μ-expansion.

Lemma 2.4 (f_μ-expansion). *Let $\mu=(a, b)\in D-\left(D_0\cup\bigcup_{k=2}^{\infty} D_3^{(k)}\right)$. If we denote by ω_μ^x the image of x under the realization π_μ, then*

$$x=1-\frac{1}{b}\sum_{n=0}^{\infty}\left(\frac{1}{a}\right)^{n-\sum_{i=1}^{n-1}\omega_\mu^x(i)}\left(-\frac{1}{b}\right)^{\sum_{i=1}^{n-1}\omega_\mu^x(i)}. \tag{9}$$

It is known that f_μ has an invariant measure which is absolutely continuous with respect to the Lebesgue measure in the case $a>1$ and $b>1$ [2]. We can prove the same result for a wider class, that is, for the case $D-\left(D_0\cup\bigcup_{k=2}^{\infty} D_3^{(k)}\right)$, and give an explicit form of the density function of the invariant measure.

Theorem 2.5 (density function of invariant measure). *Let $\mu=(a, b)\in D-\left(D_0\cup\bigcup_{k=2}^{\infty} D_3^{(k)}\right)$. Let a function $h_\mu(x)$ be*

$$h_\mu(x)=c_\mu \sum_{n=0}^{\infty}\left(\frac{1}{a}\right)^{n-\sum_{i=0}^{n-1}\omega_\mu^0(i)}\left(-\frac{1}{b}\right)^{\sum_{i=0}^{n-1}\omega_\mu^0(i)} I_{[f_\mu^n(0),1]}(x), \tag{10}$$

where c_μ is a normalizing constant. Then it follows that

(i) *h_μ is a function of bounded variation and $h_\mu \geqq 0$.*

(ii) *$h_\mu(x)dx$ is an invariant measure for f_μ, that is,*

$$\int_A h_\mu(x)dx=\int_{f_\mu^{-1}A} h_\mu(x)dx \tag{11}$$

for any Borel set $A\subset[0,1]$.

The results regarding the support of this invariant measure and the maximal period in the sense of Šarkovskii are as follows:

Theorem 2.6 (the case of even period). *If $\mu=(a,b)\in D_1$, then there exist intervals $A_0, A_1, \cdots, A_{2^n-1}$ for some $n=n(\mu)$ such that*

(i) *$f_\mu A_i=A_{i+1}$ for $0\leqq i\leqq 2^n-2$ and $f_\mu A_{2^n-1}=A_0$.*

(ii) *$f_\mu^{2^n}|A_i\cong f_{\mu'}$ (topologically conjugate) for some $\mu'\in D_6$.*

(iii) *The support of h_μ is $\bigcup_{i=0}^{2^n-1} A_i$.*

In this case, f_μ has period $2^n(2m+1)$ as the maximal period for some $m=m(\mu)$.

Theorem 2.7 (the case of odd period). *In the case of $D_2, D_5^{(k)}$ and D_6, we have,*

(i) *$h_\mu(x)>0$ a.e. on $[0,1]$.*

(ii) *The dynamical system $(f_\mu, h_\mu(x)dx)$ is weak Bernoulli.*

(iii) *f_μ has period 3 as the maximal period in the cases $D_5^{(k)}$ and D_6.*

(iv) *f_μ has period $2j+1$ as the maximal period for some $j=j(\mu)\geqq 2$ in the case D_2.*

Theorem 2.8 (the case of "islands"). *If $\mu=(a,b)\in D_4^{(k)}$, then*

(i) *there exist intervals $A_0, A_1, \cdots, A_k$ such that* (a) *$f_\mu A_i=A_{i+1}$ for $0\leqq i\leqq k-1$ and $f_\mu A_k=A_0$.* (b) *$f_\mu^{k+1}|A_i\cong f_{\mu'}$ for $\mu'=(a^{k-1}b^2, a^k b)$.* (c) *If $a+b<a^k b^3$, then μ' of* (b) *belongs to D_6 and it follows that the support of h_μ is $\bigcup_{i=0}^{k} A_i$.* (d) *If $a+b\geqq a^k b^3$, then the support of h_μ is of the form $\bigcup_{i=0}^{k}(A_i^0\cup A_i^1)$ for some subintervals A_i^0 and A_i^1 of A_i $(0\leqq i\leqq k)$.*

(ii) *The topological entropy of f_μ is equal to $\log \gamma_k$.*

(iii) *The dynamical system $(f_\mu, h_\mu(x)dx)$ is ergodic but not weakly mixing.*

In this case f_μ is chaotic, for it has period 3. But the support of h_μ is not the whole interval $[0,1]$. So we call this case "islands".

The following table summarizes the results obtained above.

	maximal period	topological entropy	support of $h\mu(x)$	erogodicity w.r.t. $h_\mu(x)dx$
$\mathring{D}_0$	2	0	there exists no a.c. invariant measure	
∂D_0	4	0	$A_0 \cup A_1$	not ergodic
$\mathring{D}_1$	$2^n(2m+1)$ for some n and m		$A_0 \cup A_1 \cup \cdots \cup A_{2^n-1}$ for some n	ergodic but not weakly mixing
∂D_1	6	$\log\sqrt{2}$	$[0, 1]$	ergodic but not weakly mixing
$\mathring{D}_2$	$2m+1$ for some $m \geqq 2$		$[0, 1]$	weak Bernoulli
∂D_2	3	$\log\frac{1+\sqrt{5}}{2}$	$[0, 1]$	weak Bernoulli
$\mathring{D}_3{}^{(k)}$	3	$\log\gamma_k$	there exists no a.c. invariant measure	
$\partial D_3{}^{(k)}$	3	$\log\gamma_k$	$A_0 \cup A_1 \cup \cdots \cup A_k$	not ergodic
$\mathring{D}_4{}^{(k)}$	3	$\log\gamma_k$	$A_0 \cup A_1 \cup \cdots \cup A_k$ or $A_0{}^0 \cup A_0{}^1 \cup \cdots \cup A_k{}^0 \cup A_k{}^1$	ergodic but not weakly mixing
$\partial D_4{}^{(k)}$	3	$\log\gamma_k$	$A_0 \cup A_1 \cup \cdots \cup A_k$	ergodic but not weakly mixing
$D_5{}^{(k)}$	3		$[0, 1]$	weak Bernoulli
D_6	3		$[0, 1]$	weak Bernoulli

In the case $a=b$, we write $f_a(\omega_a^x, h_a(x))$ for $f_\mu(\mu_\mu^x, h_\mu(x)$, respectively). We can simplify the formulae (9) and (10) as follows:

$$x=1-\sum_{n=0}^{\infty} S(n, \omega_a^x)a^{-(n+1)} \tag{9$'$}$$

$$h_a(x)=c_a \sum_{n=0}^{\infty} S(n, \omega_a^x)a^{-(n+1)}I_{[f_a^n(0),1]}(x). \tag{10$'$}$$

And we can determine the topological entropy of f_a completely.

Theorem 2.9. *The topological entropy of f_a is equal to* $\log a$ *for* $1<a\leqq 2$.

There is neither "window" nor "islands" in this case.

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