6. On the Intersection Number of the Path of a Diffusion and Chains

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(Communicated by Kôsaku YOSIDA, M. J. A., Jan. 16, 1979)

1. We are concerned with the following problem which was already considered by H. P. McKean [4] for the Brownian motion: in what manner does the path of a diffusion on a manifold wind around a fixed point or a hole asymptotically? For this purpose, we shall define a stochastic version of the intersection number. As is well-known, the usual intersection number can be represented by the integral of a differential double 1-form with singularity ([1]). Although the path of the diffusion is not smooth, we can define its intersection number with a chain by using the integral of the 1-form along the path defined in [2] (see also [3]). We then study the asymptotic behaviors of such random intersection numbers to get some solutions of the above mentioned problem.

2. Let M be a d-dimensional connected orientable Riemannian manifold with a Riemannian metric g and Δ be the Laplace-Beltrami operator corresponding to g. Let $L=\Delta/2+b$, where b is a C^∞ vector field on M. Consider the minimal diffusion process $X=(X_t, P_x)$ on M corresponding to L. For any continuous mapping $c:[0,t]\to M$, we denote by $c[0,t]$ the curve determined by c: $c[0,t]=\{c(s)\,;\,0\leqq s\leqq t\}$. We regard $c[0,t]$ as a singular 1-chain ([5]).

To define the intersection number, we prepare some notations. We principally use the notations of de Rham's book ([1]). Let $\bar{\mathscr{D}}$ be the space of square integrable currents. Set $\bar{\mathscr{D}}_1=\{T\in\bar{\mathscr{D}}\,;\,T$ is homologous to zero$\}$, $\bar{\mathscr{D}}_2=\{T\in\bar{\mathscr{D}}\,;\,T$ is cohomologous to zero$\}$ and $\mathscr{D}_3=\{T\in\bar{\mathscr{D}}\,;\,T$ is harmonic$\}$. Then $\bar{\mathscr{D}}=\bar{\mathscr{D}}_1+\bar{\mathscr{D}}_2+\mathscr{D}_3$. Let H_1, H_2, H_3 be the projections on $\bar{\mathscr{D}}_1, \bar{\mathscr{D}}_2, \mathscr{D}_3$ respectively. For any 1-current T which is continuous in mean at infinity, we define H_iT by $(H_iT,\phi)=(T,H_i\phi)$, $\phi\in C^\infty\cap\bar{\mathscr{D}}$, $i=1,2,3$. Then T can be decomposed uniquely as follows: $T=H_1T+H_2T+H_3T$. Denote by $h_i(x,y)$ the kernel of H_i, $i=1,2,3$. Let $e(x,y)=*_y h_1(x,y)$ be the adjoint form of h_1 (as 1-form of y). Then e is C^∞ if $x\neq y$. It is known that $e(x,y)$ can be written locally as follows. Let Δ be the Hodge-Kodaira's Laplacian acting on 1-forms. We can choose a domain U on which a fundamental solution $\gamma(x,y)$ for $\Delta\alpha=\beta$ exists. Let $\sigma(x,y)$ be a C^∞ function supported in $U\times U$ with (i) $0\leqq\sigma\leqq 1$, (ii)

$\sigma(x, y)=1$ on a neighborhood of the diagonal set and (iii) $\sigma(x, y)=\sigma(y, x)$. We set $\gamma_1=\sigma\gamma$. There exists a C^∞ double 1-form $\psi(x, y)$ such that $e(x, y) = d_x\delta_x *_y\gamma_1(x, y) + *_y\psi(x, y)$, $x, y \in U$, where d is the exterior differential operator and δ is the adjoint of d. See [1] for the details.

Now we shall define the intersection number $I(X[0, t], c)$ of the path of X and a C^∞ singular $(d-1)$-chain c. In the following, we assume $x_0 \notin c$. For any positive integer N, we set $\sigma_N=\inf\{t\,;\, \mathrm{dist}(X_t, \partial c) \leqq N^{-1}\}$. First we consider the case that the chain c is contained in a subdomain $U_0 \subset U$. Let f be a C^∞ function on M such that (i) $0 \leqq f \leqq 1$ and (ii) $f=1$ on U_0, $f=0$ outside U. Define $\int_{x\in X[0,t\wedge\sigma_N]} e(x, y)$ $(y \in c)$ by

$$(1)\quad \int_{x\in X[0,t\wedge\sigma_N]} e(x, y) = \delta_x *_y\gamma_1(X_{t\wedge\sigma_N}, y) - \delta_x *_y\gamma_1(X_0, y) + \int_{x\in X[0,t\wedge\sigma_N]} \{f(x) *_y\psi(x, y) + (1-f(x))e(x, y) + (f(x)-1)d_x\delta_x *_y\gamma_1\},\ P_{x_0}\text{-a.s.}$$

In the above, the second term is well-defined as the integral of 1-form along the path ([2]). The integral (1) is smooth in $y \in c$ for almost all $\omega(P_{x_0})$. So the integral $\int_{y\in c}\int_{x\in X[0,t\wedge\sigma_N]} e(x, y)$ is well-defined. Define $I_N(X[0, t], c)$ by

$$I_N(X[0, t], c) = \int_{y\in c}\int_{x\in X[0,t\wedge\sigma_N]} e(x, y) - \int_{x\in X[0,t\wedge\sigma_N]}\int_{y\in c} e(x, y),\ P_{x_0}\text{-a.s.}$$

The second term of the right hand side is also well-defined as the integral of 1-form along the path ([2]), since $\int_{y\in c} e(x, y)$ is a C^∞ 1-form in x for $x \notin \partial c$ ([1]). In the general case, we can cover the chain c by a finite number of U's on which a fundamental solution exists. By using a partition of unity, we can define $I_N(X[0, t], c)$ by the same way as above. We can show that if $x_0 \notin c$, then there exists a limit

$$I(X[0, t], c) = \lim_{N\to\infty} I_N(X[0, t], c), \qquad P_{x_0}\text{-a.s.}$$

We call the limit $I(X[0, t], c)$ the intersection number of the path of diffusion X and the $(d-1)$-chain c.

To clarify the relation between the intersection number defined above and the usual intersection number $I^*(c, c')$, we state the following approximation theorem. Let Δ_n be a subdivision of $[0, \infty)$: $0=s_{n,0}<s_{n,1}<\cdots$ with $|s_{n,k}-s_{n,k-1}|<n^{-1}$, $k=1, 2, \cdots$ (see [2]). Let X_n be a polygonal geodesic approximation of X obtained by joining $X(s_{n,k-1})$ and $X(s_{n,k})$. Then it is easy to see that $X_n[0, t]$ can be regarded as a C^∞ singular 1-chain ([5]). Therefore $I^*(X_n[0, t], c)$ is well-defined.

Theorem. *If $x \notin c$, then there exists a subsequence $\{n_k\}$ such that*

$$I^*(X_{n_k}[0, t], c) \to I(X[0, t], c) \qquad \text{as } k\to\infty,\ P_x\text{-a.s.}$$

It follows from this theorem that $I(X[0,t],c)$ has similar properties as the ordinary one:

Proposition. *$I(X[0,t],c)$ has the following properties for almost all $\omega(P_x)$.*

(i) *If $x \notin c_1 \cup c_2$, then $I(X[0,t], \lambda_1 c_1 + \lambda_2 c_2) = \lambda_1 I(X[0,t], c_1) + \lambda_2 I(X[0,t], c_2)$, $\lambda_1, \lambda_2 \in \boldsymbol{R}$, where $\lambda_1 c_1 + \lambda_2 c_2$ is a linear combination of c_1 and c_2 as $(d-1)$-chains.*

(ii) *If c is a cycle, then $I(X[0,t],c)$ depends only on the homology class of $X[0,t]$.*

(iii) *If $X[0,t] \cap c = \phi$, then $I(X[0,t],c)=0$.*

(iv) *If c is a $(d-1)$-chain with integral coefficients, then $I(X[0,t],c)$ is an integer.*

3. Throughout this section we assume that M is compact. Since M is compact, (i) there exists a unique invariant measure μ of X with $\mu(M)=1$ and (ii) the potential operator R of X is well-defined: $Rf(x) = \int_0^\infty (E_x f(X_t) - \bar{f})dt$, where $\bar{f} = \int_M f(x)\mu(dx)$ ([6]). Let $c_1, \cdots, c_k$ be a basis of $(d-1)$-dimensional homology group $H_{d-1}(M)$ of M. We consider the asymptotic behavior of the path and each c_i. We set $\alpha_i = \int_{c_i} *_y h_3(x,y)$, $i=1, \cdots, k$. Then α_i is a harmonic 1-form ($i=1, \cdots, k$). Set $f_i(x) = \alpha_i(b)(x)$. We define

$$a_i = \left(\int_M \langle \alpha_i + dRf_i, \alpha_i + dRf_i \rangle(x)\mu(dx)\right)^{1/2},$$

where $\langle\ ,\ \rangle(x)$ is the inner product of $T_x^*(M)$. Then we have the following

Theorem. (i) *For any $i=1, \cdots, k$, we have*

$$\lim_{t\to\infty, t\in Q} \frac{1}{t} I(X[0,t], c_i) = \int_M f_i(x)\mu(dx), \qquad P_x\text{-a.s.}$$

(ii) *If $\int_M f_i(x)\mu(dx)=0$, we have*

$$\overline{\lim_{t\to\infty, t\in Q}} \frac{I(X[0,t], c_i)}{\sqrt{2t\log\log t}} = -\underline{\lim_{t\to\infty, t\in Q}} \frac{I(X[0,t], c_i)}{\sqrt{2t\log\log t}} = a_i, \quad P_x\text{-a.s.}$$

As an easy consequence of this theorem, we have

Corollary. *Let M be a compact Riemannian surface with genus h. Let $(A_i, B_i)_{1\leq i\leq h}$ be a canonical homology basis. Denote by C_i the hole corresponding to (A_i, B_i), $i=1, \cdots, h$. Let α_i (or β_i) be the 1-form corresponding to A_i (or B_i). If $\int_M \alpha_i(b)(x)\mu(dx) > 0$ (or <0), then for almost all $\omega(P_x)$, the path $X[0,t]$ winds C_i infinitely often only in the positive (or negative) direction along B_i. If $\int_M \alpha_i(b)(x)\mu(dx)=0$, then for almost all ω (P_x), the path $X[0,t]$ winds C_i infinitely often in both directions along B_i. The similar result holds for β_i.*

4. In this section, we assume that $M=\boldsymbol{R}^2$. Let (x^1, x^2) be the canonical coordinate of $\boldsymbol{R}^2$. We give $\boldsymbol{R}^2$ the Riemannian metric $g_{ij}=\delta_{ij}$, $i,j=1,2$. Let $b=-x^2b(r)(\partial/\partial x^1)+x^1b(r)(\partial/\partial x^2)$, $r=((x^1)^2+(x^2)^2)^{1/2}$. We consider the diffusion X corresponding to L as before. Let us consider the intersection number $I(X[0,t],c)$, where $c=[0,\infty)$. We define this by $I(X[0,t],c)=\lim_{n\to\infty} I(X[0,t],c_n)$, where $c_n=[0,n)$. Set $\psi_t=\int_0^t r_s^{-2}ds$. Then the process $B(t)$ defined by $B(t)=\log(r(\psi^{-1}(t))/r_0)$ is a Brownian motion. Let $L(t)$ be the local time at 0 of B. Then it is easy to show that $I(X[0,t],c)$ differs from $-\frac{1}{2\pi}\int_{X[0,t]} d\theta$ by only a bounded term, where $\theta=\arg(x)$. We note that $\arg X(t)=\int_{X[0,t]} d\theta$ (see [3]). We have the following

Theorem. *Let* $x\neq 0$. (i) *If* $b\in L^1([0,\infty),rdr)$, *then*

$$\varlimsup_{t\to\infty}\frac{\arg X(t)}{L(\psi(t))}=-\varliminf_{t\to\infty}\frac{\arg X(t)}{L(\psi(t))}=\infty,\quad P_x\text{-a.s.}$$

(ii) *If* $b(r)=r^{-\beta}$, $\beta\leqq 2$, *then for any* $0<\delta<1$,

$$\lim_{t\to\infty}\frac{\arg X(t)}{L(\psi(t))^2\{\log L(\psi(t))\}^{-\delta}}=-\infty,\quad P_x\text{-a.s.}$$

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