53. *Perturbation of Domains and Green Kernels of Heat Equations. III*

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§ 1. Let Ω be a bounded domain in $\mathbf{R}^n$ with smooth boundary γ. Let $\rho(x)$ be a smooth function on γ and ν_x be the exterior unit normal vector at $x \in \gamma$. For sufficiently small $\varepsilon \geq 0$, let Ω_ε be the bounded domain whose boundary γ_ε is defined by

$$\gamma_\varepsilon = \{x + \varepsilon\rho(x)\nu_x\,;\, x \in \gamma\}.$$

Let $G_\varepsilon(x, y)$ be the Green's function of the Dirichlet boundary value problem of the Laplacian on Ω_ε. We abbreviate $G_0(x, y)$ as $G(x, y)$. Put

$$\delta^k G(x, y) = \frac{\partial^k}{\partial \varepsilon^k} G_\varepsilon(x, y)\,|_{\varepsilon=0} \qquad \text{for } k = 1, 2.$$

Put

$$\nabla_z a(z) \cdot \nabla_z b(z) = \sum_{j=1}^{n} \frac{\partial a}{\partial z_j}(z) \frac{\partial b}{\partial z_j}(z) \qquad \text{for any } a(z), b(z) \in C^\infty(\Omega).$$

By $H_1(z)$ we denote the first mean curvature of γ at z. Then, Garabedian-Schiffer [1] proved the following:

$$\begin{aligned} \delta^2 G(x, y) = &-\int_\gamma \frac{\partial G(x, z)}{\partial \nu_z} \frac{\partial G(y, z)}{\partial \nu_z} (n-1) H_1(z) \rho(z)^2 d\sigma_z \\ &+ 2 \int_\Omega \nabla_z \delta^1 G(x, z) \cdot \nabla_z \delta^1 G(y, z) dz. \end{aligned} \tag{1.1}$$

Here $\partial/\partial\nu_z$ denotes the exterior normal derivative with respect to z and $d\sigma_z$ denotes the surface element of γ.

Let $U_\varepsilon(x, y, t)$ denote the fundamental solution of the heat equation with the Dirichlet boundary condition on γ_ε. Put

$$\delta^k U(x, y, t) = \frac{\partial^k}{\partial \varepsilon^k} U_\varepsilon(x, y, t)\,|_{\varepsilon=0}$$

for $k = 1, 2$. We abbreviate $\delta^1 U(x, y, t)$ as $\delta U(x, y, t)$. In [2] and [3] the author gave explicit representation of $\delta U(x, y, t)$, that is

$$\delta U(x, y, t) = \int_0^t d\tau \int_\gamma \frac{\partial U(x, z, t-\tau)}{\partial \nu_z} \frac{\partial U(y, z, \tau)}{\partial \nu_z} \rho(z) d\sigma_z. \tag{1.2}$$

We can prove the following

Theorem 1. *For* $x, y \in \Omega$, $t > 0$

$$\begin{aligned} &\delta^2 U(x, y, t) \\ &\quad = -\int_0^t d\tau \int_\gamma \frac{\partial U(x, z, t-\tau)}{\partial \nu_z} \frac{\partial U(y, z, \tau)}{\partial \nu_z} (n-1) H_1(z) \rho(z)^2 d\sigma_z \end{aligned} \tag{1.3}$$

$$+2\int_0^t d\tau \int_\gamma \frac{\partial(\delta U)(x,z,t-\tau)}{\partial\nu_z} \frac{\partial U(y,z,\tau)}{\partial\nu_z}\rho(z)d\sigma_z.$$

By (1.2) we have the following properties of $\delta U(x,y,t)$.

$$(1.4)\qquad \begin{cases}(\partial_t-\Delta_x)\delta U(x,y,t)=0 & x,y\in\Omega,\ t>0\\ \delta U(x,y,t)=(\partial/\partial\nu_y)U(x,y,t)\rho(y) & y\in\gamma,\ x\in\Omega,\ t>0\\ \lim_{t\to+0}\delta U(x,y,t)=0 & x,y\in\Omega.\end{cases}$$

Hence the second term of the right hand side of (1.2) can be represented by

$$2\int_0^t d\tau \int_\Omega \nabla_z \delta U(x,z,t-\tau)\cdot\nabla_z\delta U(y,z,\tau)dz.$$

Let $T_r(t;\varepsilon)$ denote the trace of $U_\varepsilon(x,y,t)$ on Ω_ε which is defined by

$$T_r(t;\varepsilon)=\int_{\Omega_\varepsilon} U_\varepsilon(x,x,t)dx.$$

Put $\delta^k T_r(t)=(\partial^k/\partial\varepsilon^k)T_r(t;\varepsilon)|_{\varepsilon=0}$. We abbreviate $\delta^1 T_r(t)$ as $\delta T_r(t)$.

Let $g(t)$ and $h(t)$ be functions on $(0,\infty)$. If $\lim_{t\to+0} t^p(g(t)-h(t))=0$ for any $p=1,2,\cdots$, then we write $g(t)\simeq h(t)$.

We can prove the following

Theorem 2. *For any fixed $t>0$, $\delta^2 T_r(t)$ exists and satisfies*

$$\delta^2 T_r(t)\simeq\int_\Omega \delta^2 U(x,x,t)dx.$$

Here the integral

$$\int_\Omega \delta^2 U(x,x,t)dx$$

means the improper integral in the following sense. Let $\{\Omega_j\}_{j=1}^\infty$ be an increasing family of subdomains of Ω such that for any $j=1,2,\cdots$. $\bar{\Omega}_j$ is contained in Ω_{j+1} as a compact subset and such that $\partial\Omega_j$ is diffeomorphic to γ and $\bigcup_{j=1}^\infty\Omega_j=\Omega$. Then

$$\int_\Omega \delta^2 U(x,x,t)dx=\lim_{j\to\infty}\int_{\Omega_j}\delta^2 U(x,x,t)dx.$$

§ 2. Outline of proof. In this section, we give an outline of proof of Theorem 1 and give a proposition concerning $\delta U(x,x,t)$ which is a step to prove Theorem 2.

By the definition, we have

$$\delta^2 U(x,y,t)=\frac{\partial}{\partial\varepsilon}\delta U_\varepsilon(x,y,t)|_{\varepsilon=0},$$

so we need an explicit representation of $\delta U_\varepsilon(x,y,t)$. Fix ε. And let $\tilde{\varepsilon}$ be small real number, then there exists a function $\rho_\varepsilon(\tilde{\varepsilon},x)$ such that $\gamma_{\varepsilon+\tilde{\varepsilon}}$ can be represented uniquely as

$$\gamma_{\varepsilon+\tilde{\varepsilon}}=\{x+\tilde{\varepsilon}\rho_\varepsilon(\tilde{\varepsilon},x)\nu_x^\varepsilon\,;\ x\in\gamma_\varepsilon\},$$

where ν_x^ε is the exterior unit normal vector at $x\in\gamma_\varepsilon$. Define $\rho_\varepsilon(x)$ by $\rho_\varepsilon(x)=\lim_{\tilde{\varepsilon}\to0}\rho_\varepsilon(\tilde{\varepsilon},x)$. Then, we have

$$\delta U_\varepsilon(x,y,t)=\int_0^t d\tau\int_{\gamma_\varepsilon}\frac{\partial U_\varepsilon(x,z,t-\tau)}{\partial\nu_z^\varepsilon}\frac{\partial U_\varepsilon(y,z,\tau)}{\partial\nu_z^\varepsilon}\rho_\varepsilon(z)d\sigma_z^\varepsilon,$$

for $x, y \in \Omega$, $t>0$. See [2].

We have the following

Lemma 3. *Let* $g(\varepsilon, z)=f(\varepsilon, z+\varepsilon\rho(z)\nu_z)$ *be a function of*

$$(\varepsilon, z) \in \{(-\varepsilon_0, \varepsilon_0)\times\gamma\},$$

then

$$\begin{aligned}(2.1)\quad &\frac{\partial}{\partial\varepsilon}\int_{\gamma_\varepsilon} f(\varepsilon, w)d\sigma_w^\varepsilon \\ &=\int_\gamma f(0, z)(n-1)H_1(z)\rho(z)\,d\sigma_z+\int_\gamma \frac{\partial g}{\partial\varepsilon}(\varepsilon, z)|_{\varepsilon=0}\,d\sigma_z.\end{aligned}$$

Put $\gamma^+=\{x\in\gamma\,;\,\rho(x)\geq 0\}$ and $\gamma^-=\gamma\backslash\gamma^+$. For sufficiently small ε, we put $\gamma_\varepsilon^+=\{x+\varepsilon\rho(x)\nu_x\,;\,x\in\gamma^+\}$ and $\gamma_\varepsilon^-=\gamma_\varepsilon\backslash\gamma_\varepsilon^+$. Then, we have

$$\begin{aligned}(2.2)\quad &\frac{\partial}{\partial\varepsilon}\left(\int_0^t d\tau\int_{\gamma_\varepsilon^+}\frac{\partial U_\varepsilon(x, z, t-\tau)}{\nu_z^\varepsilon}\,\frac{\partial U_\varepsilon(y, z, \tau)}{\partial\nu_z^\varepsilon}\rho_\varepsilon(z)d\sigma_z^\varepsilon\right)\Big|_{\varepsilon=0} \\ &=\int_0^t d\tau\int_{\gamma^+}\frac{\partial U(x, z, t-\tau)}{\partial\nu_z}\,\frac{\partial U(y, z, \tau)}{\partial\nu_z}(n-1)H_1(z)\rho(z)^2d\sigma_z \\ &\quad+\int_0^t d\tau\int_{\gamma^+}\lim_{\varepsilon\to+0}\varepsilon^{-1}\left(\frac{\partial U_\varepsilon(x, z_\varepsilon, t-\tau)}{\partial\nu_{z_\varepsilon}^\varepsilon}\,\frac{\partial U_\varepsilon(y, z_\varepsilon, \tau)}{\partial\nu_{z_\varepsilon}^\varepsilon}\rho_\varepsilon(z_\varepsilon)\right. \\ &\qquad\left.-\frac{\partial U(x, z, t-\tau)}{\partial\nu_z}\,\frac{\partial U(y, z, \tau)}{\partial\nu_z}\rho(z)\right)d\sigma_z.\end{aligned}$$

Here $z_\varepsilon=z+\varepsilon\rho(z)\nu_z$.

On the other hand, for $z\in\gamma^+$ we have

$$\begin{aligned}(2.3)\quad &\lim_{\varepsilon\to+0}\varepsilon^{-1}\left(\frac{\partial U_\varepsilon(x, z_\varepsilon, t)}{\partial\nu_{z_\varepsilon}^\varepsilon}-\frac{\partial U(x, z, t)}{\partial\nu_z}\right) \\ &=\frac{\partial(\delta U)(x, z, t)}{\partial\nu_z}+\frac{\partial^2U(x, z, t)}{\partial\nu_z^2}\rho(z).\end{aligned}$$

To prove (2.2), we need the following asymptotic expansion which can be proved by using *a priori* estimates of Schauder. See [3].

$$\begin{aligned}(2.4)\quad &A(z, D)(U_\varepsilon(x, z, t)-U(x, z, t)) \\ &=\varepsilon(A(z, D)\delta U)(x, z, t)+O(\varepsilon^2),\end{aligned}$$

where $O(\varepsilon^2)$ can be taken to be uniform with respect to $z\in\gamma^+$, $t>0$. Here $A(z, D)$ is an arbitrary fixed differential operator of order 1 with $C^\infty(\bar{\Omega})$ coefficients. By (2.1) and (2.3), we have the explicit representation of the second term of the left hand side of (2.2), that is

$$\begin{aligned}(2.5)\quad &2\int_0^t d\tau\int_{\gamma^+}\frac{\partial(\delta U)(x, z, t-\tau)}{\partial\nu_z}\,\frac{\partial U(y, z, \tau)}{\partial\nu_z}\rho(z)d\sigma_z \\ &-2\int_0^t d\tau\int_{\gamma^+}\frac{\partial U(x, z, t-\tau)}{\partial\nu_z}\,\frac{\partial U(y, z, \tau)}{\partial\nu_z}(n-1)H_1(z)\rho(z)^2d\sigma_z.\end{aligned}$$

On γ^- part of the boundary, we have for $z\in\gamma^-$

$$\begin{aligned}(2.6)\quad &B(z, D)(U(x, z_\varepsilon, t)-U_\varepsilon(x, z_\varepsilon, t)) \\ &=-\varepsilon(B(z, D)\delta U_\varepsilon)(x, z_\varepsilon, t)+O(\varepsilon^2),\end{aligned}$$

for an arbitrary fixed differential operator $B(z, D)$ of order 1 with $C^\infty(\mathrm{R}^n)$ coefficients. Here $O(\varepsilon^2)$ can be taken to be uniform with respect to $z\in\gamma^-$ and $t>0$. Therefore, we get the explicit representation of

$$\frac{\partial}{\partial\varepsilon}\left\{\int_0^t d\tau\int_{\gamma_\varepsilon^-}\frac{\partial U_\varepsilon(x,z,t-\tau)}{\partial\nu_z^\varepsilon}\frac{\partial U_\varepsilon(y,z,\tau)}{\partial\nu_z^\varepsilon}\rho_\varepsilon(z)d\sigma_z^\varepsilon\right\}_{|\varepsilon=0}.$$

Summing up these facts, we have Theorem 1.

It should be remarked that our proof of Theorem 1 is different from the proof of (1.1) given by Garabedian-Schiffer. Their proof depends on the interior variational method. See [1]. Our proof is a development of the original idea of Hadamard by which he studied Hadamard's variational formula.

Proof of Theorem 2 is long, so we will only give a proposition which is important by itself. Details of proof of Theorems 1 and 2 will be given elsewhere.

Proposition 4. *For a fixed $t>0$, there exists positive constant C_μ for $\mu\in(0,1)$ such that*

$$|\delta U(x,x,t)|\le C_\mu(\mathrm{dist}\,(x,\gamma))^\mu$$

holds.

References

[1] P. R. Garabedian and M. Schiffer: Convexity of domain functionals. J. Anal. Math., 2, 281–368 (1952–53).

[2] S. Ozawa: Perturbation of domains and Green kernels of heat equations. Proc. Japan Acad., 54A, 322–325 (1978).

[3] ——: Studies on Hadamard's variational formula. Master's Thesis, Univ. of Tokyo (1979) (in Japanese).