50. *Invariants of Reflection Groups in Positive Characteristics*

By Haruhisa NAKAJIMA
Department of Mathematics, Keio University

(Communicated by Shokichi IYANAGA, M. J. A., June 12, 1979)

Let k be a field of characteristic $p>0$. Let V be a vector space over the field k and $k[V]$ be the symmetric algebra of V. Let G be a finite subgroup of $GL(V)$ with $p\,||G|$. Then G can be regarded as a subgroup of the automorphism group of $k[V]$. In [5], we classified irreducible groups G such that the invariant subrings $k[V]^G$ are polynomial rings under certain conditions. For a reflection group G, it is well known (e.g. [1]) that $k[V]^G$ is a unique factorization domain, but it has not been known whether $k[V]^G$ is a Macaulay ring. In this note we give some examples of reflection groups G such that $k[V]^G$ are not Macaulay rings.

Suppose that $p>2$ and that n is an integer with $p\,|\,n$, $n\geqq 5$. Let $E=\oplus_{i=1}^n kT_i$, $V=\oplus_{i=2}^n k(T_i-T_1)$ and $V'=V/k\sum_{i=1}^n T_i$ be vector spaces over k. The symmetric group S_n acts on $\{T_1, \cdots, T_n\}$ as permutations. Then the k-spaces V and V' are naturally regarded as S_n-faithful kS_n-modules. The group S_n is generated by reflections in $GL(V)$ and $GL(V')$ respectively. It is proved in [5] that $k[V]^{S_n}$ and $k[V']^{S_n}$ are not polynomial rings. The purpose of this note is to show the following stronger result:

Theorem. *Suppose that $p\,\|\,n$ and $p\geqq 7$. Then $k[V]^{S_n}$ and $k[V']^{S_n}$ are not Macaulay rings.*

Proof. Set $X_i=T_i-T_1$ $(2\leqq i\leqq n)$ and

$$u=\begin{bmatrix} 1 & 1 & 1\cdots 1\\ 1 & 2 & 1\cdots 1\\ 1 & 1 & 2\cdots 1\\ & \cdots\cdots & \\ 1 & 1 & 1\cdots 2\end{bmatrix}\in GL_{n-1}(k).$$

Put ${}^t[Y_2, \cdots, Y_n]=u\,{}^t[X_2, \cdots, X_n]$ and denote by $\overline{Y}_i$ $(3\leqq i\leqq n)$ the canonical images of Y_i in V'. Let

$$a={}^t[1, 2, \cdots, p-1, 0, 1, \cdots, p-1, \cdots, 0, 1, \cdots, p-1]\in k^{n-1}$$

and choose the element $a'={}^t[a'_3, \cdots, a'_n]\in k^{n-2}$ such that $ua=\begin{bmatrix}0\\ a'\end{bmatrix}$. Put

$$W_2=X_2-1,\ W_3=X_3-2,\ \cdots,\ W_p=X_p-(p-1),\ W_{p+1}=X_{p+1},$$
$$W_{p+2}=X_{p+2}-1,\ \cdots,\ W_{2p}=X_{2p}-(p-1),\ \cdots,\ W_{n-p+1}=X_{n-p+1},$$
$$W_{n-p+2}=X_{n-p+2}-1,\ \cdots,\ W_n=X_n-(p-1).$$

Let M be the maximal ideal of $k[V]$ generated by the set $\{W_2, \cdots, W_n\}$

and let M' be the maximal ideal of $k[V']$ generated by the set $\{\bar{Y}_3-a'_3, \cdots, \bar{Y}_n-a'_n\}$. We denote by H (resp. H') the decomposition group of S_n at M (resp. M') under the natural action of S_n on $k[V]$ (resp. $k[V']$). It is clear that $H'=H$.

Set $m=n/p$ and

$$B=T_2+2T_3+\cdots+(p-1)T_p+T_{p+2}+2T_{p+3}+\cdots+(p-1)T_{2p}+\cdots+T_{(m-1)p+2}+2T_{(m-1)p+3}+\cdots+(p-1)T_n.$$

For $\sigma\in H$, there is an element d of F_p such that $B^\sigma=B+dY_2$. Hence

$$H=\{\sigma\in S_n : B^\sigma=B+dY_2 \text{ for some } d\in F_p\}.$$

Let

$$\Omega=\{(i,j)\in S_n : i\equiv j \bmod p,\ i\neq j\}$$

and take the element $\tau=(1,2,\cdots,n)\in S_n$. Then H is generated by the set $\Omega\cup\{\tau\}$ and the group $J=\langle\Omega\rangle$ is a normal subgroup of H. We see that $H/J=\langle\tau J\rangle$ is a cyclic group of order p. Obviously $J=S_m\times\cdots\times S_m$ (p times). We denote by F the stabilizer subgroup of J at the set $\{X_{p+1}, X_{2p+1}, \cdots, X_{(m-1)p+1}\}$ under the natural action of J on V (i.e. the group J acts trivially on this set). Then $F\cong S_m\times\cdots\times S_m$ ($p-1$ times) and F is a normal subgroup of J.

Let $U_i(Z_1,\cdots,Z_m)$ be the fundamental symmetric polynomial of degree i with variables $Z_1,\cdots,Z_m$ and put

$$U_i^{(j)}=U_i(X_j, X_{p+j},\cdots,X_{(m-1)p+j}) \qquad (1\leqq i\leqq m;\ 2\leqq j\leqq p).$$

Then we have $k[V]^F=A[U_1^{(2)},\cdots,U_m^{(2)},\cdots,U_1^{(p)},\cdots,U_m^{(p)}]$ where $A=k[X_{p+1},X_{2p+1},\cdots,X_{(m-1)p+1}]$. Since $(p,m)=1$, we can set

$$V_i^{(j)}=\frac{1}{m}\sum_{r=0}^{m-1} U_i(T_j-T_{pr+1}, T_{p+j}-T_{pr+1},\cdots,T_{(m-1)p+j}-T_{pr+1})$$

$(1\leqq i\leqq m;\ 2\leqq j\leqq p)$. Regard $V_i^{(j)}$ as a polynomial with variables $\{X_{p+1}, X_{2p+1},\cdots,X_{(m-1)p+1}\}$, then it follows that

$$V_i^{(j)}-U_i^{(j)}\in A[U_1^{(j)},\cdots,U_{i-1}^{(j)}].$$

Hence $k[V]^F=A[V_1^{(2)},\cdots,V_m^{(2)},\cdots,V_1^{(p)},\cdots,V_m^{(p)}]$. Since $V_i^{(j)}$ $(1\leqq i\leqq m;\ 2\leqq j\leqq p)$ are contained in $k[V]^J$, we have

$$k[V]^J=A^{J/F}[V_1^{(2)},\cdots,V_m^{(2)},\cdots,V_1^{(p)},\cdots,V_m^{(p)}].$$

We identify J/F with S_m. Then S_m acts faithfully on the vector space $Q=\sum_{r=1}^{m-1} kX_{rp+1}$. Since all transpositions of S_m are represented by reflections in $GL(Q)$ and $(m,p)=1$, the module Q is kS_m-isomorphic to D/D^{S_m}, where D is the canonical representation of S_m of degree m. Hence we have the canonical epimorphism $\varphi: k[D]\to k[Q]$ which is compatible with the action of S_m. Clearly $k[D]$ is a free $k[D]^{S_m}$-module and so we know that $k[Q]$ is a free $k[Q]^{S_m}$-module by the use of the epimorphism φ. Consequently $k[Q]^{S_m}$ and $A^{J/F}$ are polynomial rings.

Put $\bar{\tau}=\tau J$ and $L_i=(\bar{\tau}-1)V_1^{(i+1)}$ $(1\leqq i\leqq p-2)$. Let I (resp. P) be the ideal of $k[V]^J$ generated by the set $(\bar{\tau}-1)k[V]^J$ (resp. the set $\{L_i : 1\leqq i\leqq p-2\}$). $k[V]^J$ is a graded polynomial subalgebra of $k[V]$. Hence

P is a prime ideal of $k[V]^J$. Since $L_1 = V_1^{(3)} - 2V_1^{(2)}$, $L_2 = V_1^{(4)} - V_1^{(3)} - V_1^{(2)}$, $\cdots$, $L_{p-2} = V_1^{(p)} - V_1^{(p-1)} - V_1^{(2)}$ are linearly independent over k, we have $ht(P) > 2$ and $\dim (k[V]^J/I) < n-3$. Let N be the homogeneous maximal ideal of $k[V]^J$ and put $R = (k[V]^J)_N$. Then we obtain depth $R^{\langle\bar{\tau}\rangle} \leqq \dim R/IR + 2 < n-1$, by Theorem 3 of [2]. Therefore $k[V]^H$ is not a Macaulay ring. If we regard $k[V] = k[W_2, \cdots, W_n]$ as a graded polynomial algebra by $deg(W_i) = 1$ $(2 \leqq i \leqq n)$, $k[V]^H$ is a noetherian graded subalgebra of $k[W_2, \cdots, W_n]$. By Proposition 4.10 of [4], $(k[V]^H)_{M \cap k[V]^H}$ is not a Macaulay ring. Since the local homomorphism

$$(k[V]^{S_n})_{M \cap k[V]^{S_n}} \rightarrow (k[V]^H)_{M \cap k[V]^H}$$

is étale, $(k[V]^{S_n})_{M \cap k[V]^{S_n}}$ is not a Macaulay ring. We conclude that $k[V]^{S_n}$ is not a Macaulay ring.

Clearly $k[V]^J/Y_2k[V]^J$ is a polynomial ring. Because the rings $k[V]^J/Y_2k[V]^J$ and $(k[V]/Y_2k[V])^J$ have the common quotient field, we get the canonical isomorphism $k[V]^J/Y_2k[V]^J \simeq k[V']^J$ which is compatible with the action of H/J. Let I' be the ideal of $k[V']^J$ generated by the set $(\bar{\tau}-1)k[V']^J$ and let P' be the ideal of $k[V']^J$ generated by the set $\{\bar{L}_i : 1 \leqq i \leqq p-2\}$, where $\bar{L}_i$ $(1 \leqq i \leqq p-2)$ are the canonical images of L_i in $k[V]^J/Y_2k[V]^J$. Since P' is a prime ideal and $ht(P') \geqq ht(P) - 1$, we have $\dim(k[V']^J/I') < n-2-2$. Therefore it follows that $k[V']^{S_n}$ is not a Macaulay ring.

Remark. The modules V and V' are naturally regarded as kA_n-modules. By Proposition 13 of [3], $k[V]^{A_n}$ and $k[V']^{A_n}$ are not Macaulay rings under the assumption of our theorem.

References

[1] A. Dress: On finite groups generated by pseudo-reflections. J. Algebra, **11**, 1–5 (1969).

[2] G. Ellingsrud and T. Skjelbred: Profondeur d'anneaux d'invariants en caractéristique p. C. R. Acad. Sci. Paris, **286**, 321–322 (1978).

[3] M. Hochster and J. Eagon: Cohen-Macaulay rings, invariant theory and the generic perfection of determinantal loci. Amer. J. Math., **93**, 1020–1058 (1971).

[4] M. Hochster and L. J. Ratliff, Jr.: Five theorems on Macaulay rings. Pacific J. Math., **44**, 147–172 (1973).

[5] H. Nakajima: Invariants of finite groups generated by pseudo-reflections in positive characteristic (to appear).