47. Some Characterization of the Schwartz Space and an Analogue of the Paley-Wiener Type Theorem on Rank 1 Semisimple Lie Groups

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In this paper we shall reform the results of Harish-Chandra [4], [5], [6] and obtain an analogue of Paley-Wiener type theorem on real rank one semisimple Lie groups.

In the first place we shall define a slightly different Fourier transform on τ-spherical functions on the Schwartz space $\mathcal{C}(G, \tau)$ and determine its image under this transform. Consequently we obtain a special case of J. Arthur's results restricted to τ-spherical functions. Next we assume that real rank of G is one. Under this assumption, we study Paley-Wiener type theorem, i.e., we shall obtain a precise description of the image of compactly supported functions in $\mathcal{C}(G, \tau)$ with respect to the above transform. The method we shall use is the same as in O. Campoli [2].

1. Notation. Let G be a real reductive Lie group with compact center and be in class $\mathcal{H}$ (cf. V. S. Varadarajan [9]). Let K be a maximal compact subgroup of G. By a parabolic pair in G we mean a pair (P, A) where P is a parabolic subgroup of G and A is its split component. Let $P=MAN$ and $\mathfrak{p}=\mathfrak{m}+\mathfrak{e}+\mathfrak{n}$ be the Langlands decomposition of P and its Lie algebra. Let $\tau=(\tau_1, \tau_2)$ be a unitary double representation of K on a finite dimensional Hilbert space V which satisfies the assumptions in Harish-Chandra [6]. Let τ_M be a representation of $K_M=K\cap M$ that is the restriction of τ to K_M. Then we can define the V-valued Schwartz space $\mathcal{C}(G, V)$ on G and the space of τ-spherical functions $\mathcal{C}(G, \tau)$ as usual. In the same way we can also define $\mathcal{C}(M, V)$ and $\mathcal{C}(M, \tau_M)$ respectively (cf. Harish-Chandra [4]). Let $\mathcal{E}_2(M)$ be the discrete series of M and $\mathcal{H}_\omega$ be the smallest closed subspace of L_2-space on M containing all matrix coefficients of ω (cf. Harish-Chandra [5]). Let $L={}^\circ\mathcal{C}(M, \tau_M)$ be the space of cusp forms on M of the type τ_M. Then $\dim L<\infty$ and L is an orthogonal sum of $L(\omega)$ for $\omega\in\mathcal{E}_2(M)$ where $L(\omega)=L\cap(\mathcal{H}_\omega\otimes V)$. Let $W=W(G/A)$ be a Weyl group of (G, A) and $W(\omega)$ be a subset of W which consists of $s\in W$ such that $s\omega=\omega$ for $\omega\in\mathcal{E}_2(M)$. Let $\mathcal{F}$ be the dual space of the Lie algebra of A. We shall regard $\mathcal{F}$ as a Euclidean space and define the Schwartz space on it as usual which denotes $\mathcal{C}(\mathcal{F})$. We define $r=r(G/A)=r(P)$,

$c=c(G/A)=c(P)$ and $\mu(\omega, \nu)$ as usual (cf. Harish-Chandra [6]).

Other notations we shall use are the same as in Harish-Chandra's papers [4], [5], [6].

2. Orthonormal basis in L. Now we fix a parabolic subgroup $P=MAN$. Then $L={}^\circ\mathcal{C}(M, \tau_M)$ can be decomposed as

$$L=\bigoplus_{1\leq j\leq m}\bigoplus_{s_j\in W-W(\omega_j)} L(s_j\omega_j) \tag{2.1}$$

where $\omega_j\in\mathcal{E}_2(M)$ for $1\leq j\leq m$.

Next we shall choose an orthonormal basis of $L(\omega_j)$ $(1\leq j\leq m)$ as follows;

$$\{\phi_i^j,\ 1\leq i\leq n_j \text{ where } n_j=\dim L(\omega_j)\} \qquad (1\leq j\leq m). \tag{2.2}$$

We denote $\phi_i^j=e_k$ where $k=n_1+n_2+\cdots+n_{j-1}+i$ $(1\leq k\leq n=n_1+n_2+\cdots+n_m)$.

3. Definition of transform. For $f\in\mathcal{C}(G,\tau)$ and $\psi\in{}^\circ\mathcal{C}(M,\tau_M)$ we define the following transform;

$$\begin{aligned}\hat f(\psi,\nu)&=(c^2r)^{-1}(f, E(P:\psi:\nu:\cdot))\\ &=(c^2r)^{-1}\int_G (f(x), E(P:\psi:\nu:x))dx \qquad (\nu\in\mathcal{F})\end{aligned} \tag{3.1}$$

where $E(P:\psi:\nu:x)$ is an Eisenstein integral and $(\,,)$ under the integral is a positive definite continuous Hermitian form on V which is invariant under τ.

Next for $\alpha\in\mathcal{C}(\mathcal{F})$ and $\psi\in{}^\circ\mathcal{C}(M,\tau_M)$ we define,

$$\check\alpha(\psi,x)=\int_F \mu(\omega,\nu)E(P:\psi:\nu:x)\alpha(\nu)d\nu \qquad (x\in G). \tag{3.2}$$

Then for fixed $\psi\in L(\omega)$, $f\mapsto\hat f$ is a continuous map of $\mathcal{C}(G,\tau)$ into $\mathcal{C}(\mathcal{F})$ and $\alpha\mapsto\check\alpha$ is a continuous map of $\mathcal{C}(\mathcal{F})$into $\mathcal{C}(G,\tau)$ (cf. Harish-Chandra [6]).

Now we shall define a slightly different Fourier transform on $\mathcal{C}(G,\tau)$ as follows;

$$E_A(f)=(\hat f(e_1,\nu),\hat f(e_2,\nu),\cdots,\hat f(e_n,\nu)) \qquad (\nu\in\mathcal{F}). \tag{3.3}$$

If we put, $E_j(f)=(\hat f(e_{k_j+1},\nu),\hat f(e_{k_j+2},\nu),\cdots,\hat f(e_{k_j+n_j},\nu))$ where $k_j=n_1+n_2+\cdots+n_{j-1}$ and $\nu\in\mathcal{F}$, then we can write E_A as

$$E_A(f)=(E_1(f),E_2(f),\cdots,E_m(f)). \tag{3.4}$$

Then it is clear that $E_A(f)$ is in $\mathcal{C}(\mathcal{F})^n$ and $E_j(f)$ is in $\mathcal{C}(\mathcal{F})^{n_j}$ for $1\leq j\leq m$.

4. Main results. Let V be an arbitrary element in $\mathcal{C}(\mathcal{F})^n$. Then V can be written as follows;

$$V=(V_1,V_2,\cdots,V_m) \tag{4.1}$$

where V_j is an element in $\mathcal{C}(\mathcal{F})^{n_j}$ $(1\leq j\leq m)$. Now we shall define two subspaces of $\mathcal{C}(\mathcal{F})^n$. Let $\mathcal{C}(\mathcal{F})^n_*$ denote the closed subspace of $\mathcal{C}(\mathcal{F})^n$ consisting of elements which satisfy the following relation,

$$V_j(s^{-1}\nu)^t=\overline{{}^\circ C_{P|P}(s:s^{-1}\nu)}V_j(\nu)^t \quad \text{for all } s\in W(\omega_j)\ (1\leq j\leq m) \text{ and } \nu\in\mathcal{F}, \tag{4.2}$$

where V_j^t is the transposed vector of V_j $(1 \leq j \leq m)$ and ${}^\circ C_{P|P}(s; s^{-1}\nu)$ is a unitary operator which maps $L(\omega)$ onto $L(s\omega)$ $(s \in W)$ (cf. Harish-Chandra [6]). We regard this operator as a matrix operator on $L(\omega)$ with respect to the basis in (2.2) and denote the complex conjugate by $\overline{\quad}$.

Let $\mathcal{H}(\mathcal{F})_*^n$ denote the subspace of $\mathcal{C}(\mathcal{F})_*^n$ consisting of elements V whose each component v_i^j $(V_j=(v_1^j, v_2^j, \cdots, v_{n_j}^j),\ 1 \leq j \leq m)$ extends to the holomorphic function which is an exponential type and satisfies the following condition; if there exists a relation,

$$\sum_{i,j,t} C_{i,j,t}\left(\frac{d^{m_t}}{d\nu^{m_t}}\right)\bigg|_{\nu=\nu_t} E(P:\phi_i^j:\nu:x)=0 \tag{4.3}$$

where m_t is a non-negative integer, $\nu_t \in (\mathcal{F})_c$ (the complexification of $\mathcal{F}$) and $C_{i,j,t}$ is in $\boldsymbol{C}$, then

$$\sum_{i,j,t} C_{i,j,t}\left(\frac{d^{m_t}}{d\nu^{m_t}}\right)\bigg|_{\nu=\nu_t} v_i^j(\nu)=0. \tag{4.4}$$

Now we shall decompose $\mathcal{C}(G, \tau)$. Let $\Gamma_1, \Gamma_2, \cdots, \Gamma_r$ be a complete set of θ-stable Cartan subgroups of G, no two of which are conjugate and put $A_i=(\Gamma_i)_R$ (the vector part of Γ_i) for $1 \leq i \leq r$. Let $P_i=M_iA_iN_i$ be a parabolic subgroup whose split component is A_i $(1 \leq i \leq r)$. Let $\mathcal{C}_i(G, \tau)$ denote the closed subspace of $\mathcal{C}(G, \tau)$ consisting of all f satisfying $f^{(P')} \sim 0$ unless A' $(P'=M'A'N')$ is conjugate to A_i under$K$. Then $\mathcal{C}(G, \tau)$ is decomposed as follows (cf. Harish-Chandra [6]);

$$\mathcal{C}(G, \tau)=\mathcal{C}_1(G, \tau)\oplus\mathcal{C}_2(G, \tau)\oplus\cdots\oplus\mathcal{C}_r(G, \tau) \tag{4.5}$$

(topological direct sum). When we apply the above consideration to P_i $(1 \leq i \leq r)$, we shall use the notation such that E_{A_i}, $\mathcal{F}_i$ and $n^{(i)}$ instead of E_A, $\mathcal{F}$ and n. When Γ_i is a compact Cartan subgroup, then $\mathcal{C}_i(G, \tau)$ coincides with ${}^\circ\mathcal{C}(G, \tau)$ and E_{A_i} is the identity mapping.

Theorem 1. *If Γ_i is not compact, then the mapping E_{A_i} is a homeomorphism of $\mathcal{C}_i(G, \tau)$ onto $\mathcal{C}(\mathcal{F}_i)_*^{n^{(i)}}$.*

$$\begin{array}{ccccccc} \mathcal{C}(G, \tau)= & \mathcal{C}_1(G, \tau) & \oplus & \mathcal{C}_2(G, \tau) & \oplus\cdots\oplus & \mathcal{C}_r(G, \tau) \\ & \downarrow E_{A_1} & & \downarrow E_{A_2} & & \downarrow E_{A_r} \\ & \mathcal{C}(\mathcal{F}_1)_*^{n^{(1)}} & \oplus & \mathcal{C}(\mathcal{F}_2)_*^{n^{(2)}} & \oplus\cdots\oplus & \mathcal{C}(\mathcal{F}_r)_*^{n^{(r)}} \end{array}$$

Theorem 2. *Assume that the real rank of G is equal to one. Then the mapping E_A is a homeomorphism of $\mathcal{C}_A(G, \tau)$ onto $\mathcal{C}(\mathcal{F})_*^n$. An element V in $\mathcal{C}(\mathcal{F})_*^n$ belongs to $\mathcal{H}(\mathcal{F})_*^n$ if and only if there exists a function f in $C_c^\infty(G, \tau)$ such that $V=E_A(f)$. If rank $K \neq$* rank G, *then ${}^\circ\mathcal{C}(G, \tau)$ $=0$.*

$$\begin{array}{ccc} \mathcal{C}(G, \tau)={}^\circ\mathcal{C}(G, \tau) & \oplus & \mathcal{C}_A(G, \tau) \\ & & \downarrow E_A \\ & & \mathcal{C}(\mathcal{F})_*^n \end{array}$$

Remark 1. In Theorem 1 we have a following inversion formula for $f \in \mathcal{C}(G, \tau)$;

$$f(x)=\sum_{p=1}^{r}\sum_{j^{(p)}=1}^{m^{(p)}}|W^{(p)}(\omega^{(p)}_{j^{(p)}})|^{-1}\sum_{i^{(p)}=1}^{n^{(p)}_{j^{(p)}}}\int_{\mathcal{F}_p}\mu^{(p)}(\omega^{(p)}_{j^{(p)}},\nu_p)$$
$$\times E(P_p\,;\phi^{j^{(p)}}_{i^{(p)}}:\nu_p\,;x)\hat{f}(\phi^{j^{(p)}}_{i^{(p)}},\nu_p)d\nu_p.$$

Remark 2. In Theorem 2 if we put $\tau_1=\tau_2=$trivial representation and $V=\mathcal{C}$, then Theorem 2 coincides with the result of S. Helgason [7] and R. Gangolli [3]. In the same way if we put $\tau_1=$trivial representation and $\tau_2=$arbitrary, then Theorem 2 coincides with a theorem of S. Helgason [8].

Remark 3. In Theorem 2 we can obtain a relation between a size of a support of a compactly supported function and an exponential type of its Fourier transform as usual.

Remark 4. Using Theorem 2, we have obtained Paley-Wiener type theorem on $\mathcal{C}(G)$, when real rank of G is one. We shall describe this result in a next article.

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