45. On a Nature of Convergence of Some Feynman Path Integrals. I

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§1. **Introduction.** In the previous papers [6], [7] and [8], we discussed convergence in the uniform operator topology of the Feynman path integral under some assumptions concerning the potential function. In this note, we shall discuss pointwise convergence of the Feynman path integral and we shall prove that it converges in a very strong topology if the potential function satisfies the same assumptions as in the previous papers [6], [7] and [8]. As to the notion of the Feynman path integral we refer Feynman [4] and Feynman-Hibbs [5].

Let $x=(x_1, x_2, \cdots, x_n)$ denote a point of $\mathbf{R}^n$. We shall treat the quantum dynamical system described by the Lagrangean of the form

$$L(t, x, \dot{x})=\frac{1}{2}|\dot{x}|^2-V(t, x). \tag{1}$$

The potential function $V(t, x)$ is assumed to satisfy the following assumptions;

(A-I) *$V(t, x)$ is a real-valued function of $(t, x)\in \mathbf{R}\times\mathbf{R}^n$. For any fixed $t\in\mathbf{R}$, $V(t, x)$ is a function of $x\in\mathbf{R}^n$ of class C^∞. $V(t, x)$ is a measurable function of $(t, x)\in\mathbf{R}\times\mathbf{R}^n$.*

(A-II) *For any multi-index α with its length $|\alpha|\geq 2$, the nonnegative measurable function of t defined by*

$$M_\alpha(t)=\sup_{x\in\mathbf{R}^n}\left|\left(\frac{\partial}{\partial x}\right)^\alpha V(t, x)\right|+\sup_{|x|\leq 1}|V(t, x)| \tag{2}$$

is essentially bounded on every compact interval of $\mathbf{R}$.

We fix a large integer K, say, $K=100(n+100)$. We put $T=\infty$ if $\sum_{2\leq|\alpha|\leq K}\text{ess. sup}_{t\in\mathbf{R}} M_\alpha(t)<\infty$. Otherwise, we let T denote an arbitrary fixed positive number. We shall discuss everything in the time interval $(-T, T)$.

Let $S(t, s, x, y)$ be the classical action along the classical orbit starting from y at time s and reaching x at time t. We can prove that there exists a positive constant $\delta_1(T)$ such that $S(t, s, x, y)$ is uniquely defined for any x and y in $\mathbf{R}^n$ if $|t-s|\leq\delta_1(T)$(cf. [4, Proposition 1]).

We shall consider the following integral transformation:

$$E^{(0)}(\lambda, t, s)\varphi(x)=\left(\frac{-\lambda}{2\pi(t-s)}\right)^{n/2}\int_{\mathbf{R}^n} e^{\lambda S(t,s,x,y)}\varphi(y)dy, \tag{3}$$

where $\lambda=i\hbar^{-1}$, $\hbar$ being a small positive parameter (=the Planck constant). This integral transformation was used in [7] and [8] (see also [4]). Let $[s, t]$ be an arbitrary time interval contained in $(-T, T)$ and let

$$\Delta\,;\, s=t_0<t_1<t_2<\cdots<t_L=t \tag{4}$$

be an arbitrary subdivision of the interval $[s, t]$. We denote

$$\delta(\Delta)=\max_{1\le j\le L}|t_j-t_{j-1}|. \tag{5}$$

We treat the iterated integral

$$\begin{aligned} &I^{(0)}(\Delta\,|\,\lambda, t, s, x, y)\\ &=\prod_{j=1}^{L}\left(\frac{-\lambda}{2\pi(t_j-t_{j-1})}\right)^{n/2}\\ &\quad\times\int_{\mathbf{R}^n\times\cdots\times\mathbf{R}^n}\cdots\int \exp \lambda\sum_{j=1}^{L} S(t_j, t_{j-1}, x^j, x^{j-1})dx^1\cdots dx^{L-1}.\end{aligned} \tag{6}$$

Here x^0 and x^L denote y and x, respectively. We shall prove that $I^{(0)}(\Delta\,|\,\lambda, t, s, x, y)$ converges as $\delta(\Delta)$ tends to 0. More precise statements will be found in the next section.

§2. **Main results.** We apply a version of the stationary phase method presented in [1] to the integral (6). Then, we can prove

Proposition 1. *Assume that* $|t-s|\le\delta_1(T)$. *Then, we can write*

$$\begin{aligned} &I^{(0)}(\Delta\,|\,\lambda, t, s, x, y)\\ &=\left(\frac{-\lambda}{2\pi(t-s)}\right)^{n/2} a^{(0)}(\Delta\,|\,\lambda, t, s, x, y)e^{\lambda S(t,s,x,y)}.\end{aligned} \tag{7}$$

The amplitude function $a^{(0)}(\Delta\,|\,\lambda, t, s, x, y)$ *belongs to the function space* $\mathcal{B}(\mathbf{R}^n_x\times\mathbf{R}^n_y)$ *of Schwartz* [11] *as a function of* $(x, y)\in\mathbf{R}^n\times\mathbf{R}^n$ *with parameters* Δ, λ, t *and* s.

We shall denote the above function $a^{(0)}(\Delta\,|\,\lambda, t, s, x, y)$ by $a^{(0)}(\Delta\,|\,\lambda, t, s)$ when we consider it as an element of the function space $\mathcal{B}(\mathbf{R}^n\times\mathbf{R}^n)$ with parameters Δ, λ, t and s. As usual, we shall use the norm $\|\ \|_m$ defined by

$$\|f\|_m=\sum_{|\alpha|+|\beta|\le m}\sup_{(x,y)\in\mathbf{R}^n\times\mathbf{R}^n}\left|\left(\frac{\partial}{\partial x}\right)^{\alpha}\left(\frac{\partial}{\partial y}\right)^{\beta}f(x, y)\right|. \tag{8}$$

Our main result is

Theorem 1. *Assume that* $|t-s|\le\delta_1(T)$. *Then, the limit*

$$k(\lambda, t, s)=\lim_{\delta(\Delta)\to 0} a^{(0)}(\Delta\,|\,\lambda, t, s) \tag{9}$$

exists in the function space $\mathcal{B}(\mathbf{R}^n_x\times\mathbf{R}^n_y)$. *Moreover, for any positive integer* m, *there exist positive constants* $\gamma_1(m, T)$ *and* $\delta_2(T)$ *such that we have*

$$\|k(\lambda, t, s)-a^{(0)}(\Delta\,|\,\lambda, t, s)\|_m\le\gamma_1(m, T)\delta(\Delta), \tag{10}$$

for any subdivision Δ *with* $\delta(\Delta)\le\delta_2(T)$. *The constants* $\gamma_1(m, T)$ *and* $\delta_2(T)$ *are independent of any particular choice of* t, s, Δ *and* λ *if* $|\lambda|$ *is bounded away from* 0. $\delta_2(T)$ *is independent of* m.

We can make the convergence of the iterated integral faster if we replace the parametrix $E^{(0)}(\lambda, t, s)$ by $E^{(N)}(\lambda, t, s)$, $N=1, 2, 3\cdots$ of Fujiwara [6] and [8]. Let N be any fixed positive integer and let

$$a^{(N)}(\lambda, t, s, x, y)=\sum_{j=1}^{N} \lambda^{1-j} a_j(t, s, x, y) \tag{11}$$

be the function (11) of [6] with N replaced by $N-1$. We use the integral transformation with the oscillatory kernel

$$E^{(N)}(\lambda, t, s)\varphi(x)=\left(\frac{-\lambda}{2\pi(t-s)}\right)^{n/2} \int_{\mathrm{R}^n} a^{(N)}(\lambda, t, s, x, y)e^{\lambda S(t,s,x,y)}\varphi(y)dy \tag{12}$$

in place of (3). We use this and make the iterated integral

$$\begin{aligned} &I^{(N)}(\Delta \mid \lambda, t, s, x, y) \\ &=\prod_{j=1}^{L}\left(\frac{-\lambda}{2\pi(t_j-t_{j-1})}\right)^{n/2} \int_{\mathrm{R}^n\times\cdots\times\mathrm{R}^n}\cdots\int \prod_{j=1}^{L} a^{(N)}(\lambda, t_j, t_{j-1}, x^j, x^{j-1}) \\ &\quad \exp \lambda \sum_{j=1}^{L} S(t_j, t_{j-1}, x^j, x^{j-1})dx^1\cdots dx^{L-1}. \end{aligned} \tag{13}$$

Then we have

Theorem 2. *Assume that* $|t-s|\le\delta_1(T)$. *Then, we can write*

$$I^{(N)}(\Delta \mid \lambda, t, s, x, y)=\left(\frac{-\lambda}{2\pi(t-s)}\right)^{n/2} a^{(N)}(\Delta \mid \lambda, t, s, x, y)e^{\lambda S(t,s,x,y)} \tag{14}$$

with some function $a^{(N)}(\Delta \mid \lambda, t, s) \in \mathscr{B}(\mathrm{R}^n_x\times\mathrm{R}^n_y)$. *Moreover, for any integer* $m\ge 0$, *there exists positive constant* $\gamma_2(m, T)$ *such that we have*

$$\|k(\lambda, t, s)-a^{(N)}(\Delta \mid \lambda, t, s)\|_m\le\gamma_2(m, T)\,|\lambda|^{-N}\delta(\Delta)^N, \tag{15}$$

if $\delta(\Delta)\le\delta_2(T)$. *The constant* $\gamma_2(m, T)$ *is independent of any particular choice of* t, s, Δ *and* λ *if* $|\lambda|$ *is bounded away from* 0.

The function

$$K(\lambda, t, s, x, y)=\left(\frac{-\lambda}{2\pi(t-s)}\right)^{n/2} k(\lambda, t, s, x, y)e^{\lambda S(t,s,x,y)} \tag{16}$$

is the fundamental solution (Green's function) for the Schrödinger equation. In fact putting

$$U(\lambda, t, s)\varphi(x)=\int_{\mathrm{R}^n} K(\lambda, t, s, x, y)\varphi(y)dy, \tag{17}$$

we can prove that $U(\lambda, t, s)$ coincides with the fundamental solution constructed in [6], [7] and [8].

Concerning the asymptotic behaviour of $k(\lambda, t, s, x, y)$ as $|\lambda|\to\infty$ (quasi-classical limit), we have

Theorem 3. *If* $|t-s|\le\delta_1(T)$, *then, for any integer* $m\ge 0$, *we have*

$$\left\|k(\lambda, t, s)-\sum_{j=1}^{N} \lambda^{1-j}a_j(t, s)\right\|_m \le C_m\,|\lambda|^{-N}\,|t-s|^N \tag{18}$$

with some constant C_m *depending on* m. *Here* $a_j(t, s, x, y)$ *is the function appeared in* (11).

§3. Sketch of the proof of Theorems 1 and 2. We represent the fundamental solution $U(\lambda, t, s)$ as

$$U(\lambda, t, s)=E^{(N)}\sharp F(\lambda, t, s)+E^{(N)}(\lambda, t, s), \tag{19}$$

where we used the abbreviation

(20) $$E^{(N)}\# F(\lambda, t, s)=\int_s^t E^{(N)}(\lambda, t, \sigma)F(\lambda, \sigma, s)d\sigma.$$

The operator $F(\lambda, t, s)$ should satisfy the equation

(21) $$F(\lambda, t, s)+\lambda G^{(N)}\# F(\lambda, t, s)+\lambda G^{(N)}(\lambda, t, s)=0,$$

where $G^{(N)}(\lambda, t, s)$ is the integral operator given by

(22) $$G^{(N)}(\lambda, t, s)\varphi(x)=\left(\frac{\partial}{\lambda\partial t}+\frac{1}{2}\sum_j\left(\frac{\partial}{\lambda\partial x_j}\right)^2+V(t, x)\right)E^{(N)}(\lambda, t, s)\varphi(x),$$

for any φ in $C_0^\infty(\mathrm{R}^n)$. We solve the equation (21) and obtain, at least formally,

(23) $$\begin{aligned}F(\lambda, t, s)=&-\lambda G^{(N)}(\lambda, t, s)+(-\lambda)^2G^{(N)}\# G^{(N)}(\lambda, t, s)\\&+(-\lambda)^3G^{(N)}\# G^{(N)}\# G^{(N)}(\lambda, t, s)+\cdots.\end{aligned}$$

Using the function $\omega(t, s, x, y)=(t-s)^{-1}(S(t, s, x, y)-2^{-1}(t-s)^{-1}|x-y|^2)$, we have, in [7] and [8], that

(24) $$\begin{aligned}&G^{(0)}(\lambda, t, s)\varphi(x)\\&=\left(\frac{-\lambda}{2\pi(t-s)}\right)^{n/2}\frac{(t-s)}{2\lambda}\int_{\mathrm{R}^n}\Delta\omega(t, s, x, y)e^{\lambda S(t,s,x,y)}\varphi(y)dy,\end{aligned}$$

and

(25) $$\begin{aligned}&G^{(N)}(\lambda, t, s)\varphi(x)\\&=\left(\frac{-\lambda}{2\pi(t-s)}\right)^{n/2}\frac{\lambda^{-N-1}}{2}\int_{\mathrm{R}^n}\Delta a_N(t, s, x, y)e^{\lambda S(t,s,x,y)}\varphi(y)dy.\end{aligned}$$

If $|t-s|\leq\delta_1(T)$, then we can write the kernel function of $(-\lambda)^kG^{(N)}\# G^{(N)}\#\cdots\# G^{(N)}$ as

(26) $$\left(\frac{-\lambda}{2\pi(t-s)}\right)^{n/2}c_k^{(N)}(\lambda, t, s, x, y)e^{\lambda S(t,s,x,y)}.$$

Using the fundamental lemma which we shall state in § 4, we can prove

Proposition 2. *For any integer $m\geq 0$, there exists positive constant $\gamma(m, T, N)$ such that we have*

(27) $$\|c_k^{(0)}(\lambda, t, s)\|_m\leq\frac{\gamma(m, T, 0)^k}{\Gamma(2k)}|t-s|^{2k-1},\qquad \textit{for } k\geq 2$$

and

(28) $$\|c_k^{(N)}(\lambda, t, s)\|_m\leq\frac{\gamma(m, T, N)^k}{\Gamma(k(N+2))}|\lambda|^{1-k(N+1)}|t-s|^{k(N+2)-1}$$

for $k\geq 2$ and $N\geq 1$ if $|t-s|\leq\delta_2(T)$.

This proposition implies that the infinite series

(29) $$k(\lambda, t, s, x, y)=\sum_{j=1}^\infty c_j^{(N)}(\lambda, t, s, x, y)$$

converges in the space $\mathcal{B}(\mathrm{R}_x^n\times\mathrm{R}_y^n)$ if $|t-s|\leq\delta_2(T)$. This time interval may be very shoot. But this restriction can be removed by the evolution property of the fundamental solution. We have from (23)

Proposition 3. *The function $k(\lambda, t, s, x, y)e^{\lambda S(t,s,x,y)}$ is the fundamental solution for the Schrödinger equation. For any integer $m\geq 0$,*

(30) $$\|k(\lambda, t, s)-1\|_m\leq C_m|t-s|^2$$

and

$$\|k(\lambda, t, s)-a^{(N)}(\lambda, t, s)\|_m \leq C_m |\lambda|^{-N} |t-s|^{N+1} \tag{31}$$

hold, where C_m is a positive constant independent of t, s and λ if $|\lambda|$ is bounded away from 0.

Theorems 1 and 2 follow from this and the fundamental lemma.

§ 4. The fundamental lemma. Assume that $|t-s| \leq \delta_1(T)$. Let

$$\Delta\,;\, s=t_0<t_1<\cdots<t_L=t \tag{32}$$

be an arbitrary subdivision of the interval $[s, t]$. Let $a_1(x, y)$, $a_2(x, y)$ $\cdots a_L(x, y)$ be arbitrary functions in the space $\mathscr{B}(\mathrm{R}^n \times \mathrm{R}^n)$. Consider the iterated integral

$$\begin{aligned} &\prod_{j=1}^{L}\left(\frac{-\lambda}{2\pi(t_j-t_{j-1})}\right)^{n/2} \\ &\times \int_{\mathrm{R}^n\times\cdots\times\mathrm{R}^n}\cdots\int \prod_{j=1}^{L} a_j(x^j, x^{j-1}) \exp \lambda \sum_{j=1}^{L} S(t_j, t_{j-1}, x^j, x^{j-1}) dx^1 \cdots dx^{L-1}. \end{aligned} \tag{33}$$

Then, the stationary phase method proves that this is equal to

$$\left(\frac{-\lambda}{2\pi(t-s)}\right)^{n/2} a(\Delta \,|\, \lambda, t, s, x, y) e^{\lambda S(t,s,x,y)} \tag{34}$$

with some amplitude function $a(\Delta \,|\, \lambda, t, s, x, y)$ in $\mathscr{B}(\mathrm{R}^n_x \times \mathrm{R}^n_y)$.

Lemma 4. *For any integer $m \geq 0$, there exist positive constants $\delta_2(T)$, $\kappa(m)$ and $R(m)$ such that*

$$\|a(\Delta \,|\, \lambda, t, s)\|_m \leq \kappa(m)^L \prod_{j=1}^{L} \|a_j\|_{R(m)} \tag{35}$$

if $|t-s| \leq \delta_2(T)$. The constants $\delta_2(T)$, $\kappa(m)$ and $R(m)$ are independent of L, functions $a_1, a_2, \cdots, a_L$, subdivision Δ and of λ if $|\lambda|$ is bounded away from 0. *$\delta_2(T)$ is also independent of m.*

This lemma is proved by using our previous results in [1] and a variation of the technique of Taniguchi and Kumanogo in Kumanogo [10].

Remark. Chazarain [2] [3] and Kitada [9] use parametrices similar to ours (3) and (12).

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