43. A Note on Siegel's Zeros

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(Communicated by Kunihiko KODAIRA, M. J. A., May 12, 1979)

1. Let χ be a real primitive Dirichlet character (mod q), and $L(s,\chi)$ the L-function attached to χ. Then, it may happen that $L(s,\chi)$ has a real zero $1-\delta$ such that

$$0<\delta\leqq c_1(\log q)^{-1},$$

where and also in the sequel c's are absolute constants. This, if exists, is the Siegel zero of $L(s,\chi)$.

On the other hand, let $\pi(x\,;q,l)$ be as usual the number of primes less than x and congruent to l (mod q). Then, as is well-known, the hypothetical estimate

$$\pi(x\,;q,l)\leqq(2-\xi)\frac{x}{\varphi(q)\log x/q}, \tag{1}$$

where $\xi>0$ is an absolute constant, and $\varphi(q)$ is the Euler function, implies

$$\delta\geqq c_2\xi(\log q\log\log q)^{-1} \tag{2}$$

(cf. [3]). It seems that as far as we appeal to the prime number theorem of Rodosskii and Tatuzawa [6, p. 314] the above result (2) is the best that can be deduced from (1).

2. The purpose of this short paper is to show the result which appears to be ultimately the best possible one deducible from (1):

Theorem. *If* (1) *holds for* $x\geqq q^{c_3}$, *then we have*

$$\delta\geqq c_4\xi(\log q)^{-1}.$$

Proof. There are two ways to prove this. One is via the prime number theorem of Linnik-Fogels-Gallagher [2] (see also [4]). The other one, which we are going to show below, is closely related to the Deuring-Heilbronn phenomenon,*) and much more elementary and direct.

Now, let us put

$$B(n)=\sum_{d|n}\chi(d)d^{-\delta}$$

which is non-negative for all n. And let us apply the Selberg sieve to the sequence $\{B(n)\}$. That is, we consider the expression

$$I(N,z)=\sum_{n\leqq N}B(n)\Big(\sum_{d|n}\lambda_d\Big)^2,$$

*) This fact will be analysed in our forthcoming paper in a wider context including large sieve extensions of (1) (cf. [1] [5]).

where $\lambda_1=1$ and $\lambda_d=0$ for $d>z$. This has been already investigated in [4, Section 4], according to which we have, for the optimal $\{\lambda_d\}$,

$$I(N, z)\leqq c_5\delta N, \tag{3}$$

provided, say,

$$N\geqq q^{c_6}, \qquad N^{1/4}\geqq z\geqq q^{c_7}.$$

On the other hand, denoting primes by p, we have

$$I(N, N^{1/4})\geqq \sum_{N^{1/2}<p\leqq N} B(p).$$

This sum is obviously

$$\geqq \pi(N)-\pi(N^{1/2})-\sum_{\substack{p\leqq N\\ \chi(p)=-1}} 1,$$

and thus, by (1), we have

$$\begin{aligned} I(N, N^{1/4})&>(1-o(1))\frac{N}{\log N}\\ &\quad-\frac{\varphi(q)}{2}(2-\xi)\frac{N}{\varphi(q)\log N/q}>\frac{\xi}{3}\,\frac{N}{\log N}, \end{aligned} \tag{4}$$

provided $N\geqq q^{c_8}$ with a sufficiently large c_8. From (3) and (4) the assertion of the theorem follows immediately.

Our theorem states, in other words, that (1) implies the nonexistence of the Singel zero (mod q). Thus we get readily (cf. [2][4]).

Corollary. *If* (1) *holds for* $x\geqq q^{c_3}$, *then we have*

$$\pi(x\,;\,q, l)=\left\{1+O\left(\exp\left(-c_9\xi\frac{\log x}{\log q}\right)\right)\right\}\frac{x}{\varphi(q)\log x},$$

where the constant implied by O depends on ξ effectively.

So, in particular, once we have (1), the constant $2-\xi$ will be automatically reduced to $1+\xi$ for x larger than a sufficiently high power of q. And it may be worth remarking that our corollary suggests the reason why (1) is so notoriously difficult.

Remark. An interesting result as well as a detailed history on the present problem can be found in the recent paper [7] of Siebert. He has extended the situation so as to include the Jurkat-Richert sieve estimate applied to the arithmetic progressions (mod q), in place of (1). It should be remarked that the extension of our theorem into the direction similar to that of Siebert is quite possible.

Acknowledgement. The present author is indebted to Dr. H. Siebert (Uni. Ulm) for sending him a copy of the unpublished material [7].

References

[1] E. Bombieri and H. Davenport: On the large sieve method. Number Theory and Analysis (Papers in Honour of E. Landau), New York, pp. 9–22 (1969).

[2] P. X. Gallagher: A large sieve density estimate near $\sigma=1$. Invent. Math., **11**, 329–339 (1970).

[3] Y. Motohashi: On some improvements of the Brun-Titchmarsh theorem. J. Math. Soc. Japan, **26**, 306–323 (1974).
[4] ——: Primes in arithmetic progressions. Invent. Math., **44**, 163–178 (1978).
[5] ——: Large sieve extensions of the Brun-Titchmarsh theorem (to appear).
[6] K. Pracher: Primzahlverteilung. Springer (1957).
[7] H. Siebert: Sieve methods and Siegel's zeros (to appear).