41. A Remark on the Hadamard Variational Formula

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§ 1. **Introduction.** Let $f(x)$ be a real valued C^∞ function of x in $\boldsymbol{R}^2$. Using this and real number $t \in \boldsymbol{R}$, we define the open set $\Omega_t = \{x \in \boldsymbol{R}^2 \mid f(x) < t\}$. Its boundary is $\gamma_t = \{x \in \boldsymbol{R}^2 \mid f(x) = t\}$. We assume the following assumptions for $f(x)$;

(A. 1) *Ω_1 is a non empty simply connected bounded domain in $\boldsymbol{R}^2$.*

(A. 2) *All the $t \in [-1, 0) \cup (0, 1]$ are regular values of f.*

(A. 3) *Ω_1 contains only one critical point x^0 of f. At this point, the function $f(x^0)$ has its value* 0 *and it has non-degenerate Hessian of signature of type* (1, 1).

We shall consider the Green function $g_t(x, y)$ for the Dirichlet problem in the open set Ω_t for any $t \in [-1, 1]$, that is, $g_t(x, y)$ is the solution for the following boundary value problem;

$$(1) \qquad -\Delta g_t(x, y) = \delta(x-y) \qquad \text{for any } x, y \text{ in } \Omega_t.$$

and

$$(2) \qquad g_t(x, y) = 0, \qquad \text{if } x \in \gamma_t,\ y \in \Omega_t.$$

When t decreases from 1 to any $\varepsilon > 0$, the open set Ω_t shrinks to Ω_ε. Throughout this process Ω_t is a simply connected domain with its smooth boundary, because (A. 2) and (A. 3) hold. See, for example Milnor [6]. Therefore, the celebrated Hadamard variational formula implies that $(d/dt)g_t(x, y)$ exists for $t \neq 0$ and for any x and y in Ω_t and that

$$(3) \qquad \frac{d}{dt} g_t(x, y) = \int_{\gamma_t} \frac{\partial g_t(x, z)}{\partial \nu_z} \frac{\partial g_t(y, z)}{\partial \nu_z} \frac{1}{|\operatorname{grad} f(z)|} d\sigma_z,$$

where $d\sigma_z$ is the line element of γ_t and ν_z is the unit outer normal to γ_t at z. (See Hadamard [5], Garabedian [4], Garabedian-Schiffer [3]. Simpler proof is given in Fujiwara-Ozawa [2].) This enables us to write

$$(4) \qquad g_1(x, y) - g_\varepsilon(x, y) = \int_\varepsilon^1 \frac{d}{dt} g_t(x, y) dt$$

for any $x \neq y$ in Ω_ε if $\varepsilon > 0$. Hence the following natural question arises.

(Q) *Can one replace ε in* (4) *by* -1?

This does not seem a trivial problem because the open set Ω_t has two connected components for $t \leq 0$ while it is connected for $t > 0$. The aim of this note is to prove the following affirmative answer to this question (Q).

Theorem 1. *For any $x \neq y$ in Ω_{-1}, we have*

$$\int_{-1}^{1}\left|\frac{d}{dt}g_t(x, y)\right|dt<\infty \tag{5}$$

and

$$g_1(x, y)-g_{-1}(x, y)=\int_{-1}^{1}\frac{d}{dt}g_t(x, y)dt. \tag{6}$$

§2. **Green functions.** We begin with the following four well known facts. (See, for example, Courant-Hilbert [1].)

Proposition 1. *$g_t(x, y)\geq 0$ for any x and y in Ω_t and for any t in* $[-1, 1]$.

Proposition 2. *$g_t(x, y)\leq g_{t'}(x, y)$ if $t\leq t'$ and $x, y\in\Omega_t$.*

Proposition 3. *$g_t(x, y)$ is a continuous function of x in $\bar{\Omega}_t\setminus\{y\}$.*

Proposition 4. *In the case $t\leq 0$, we have*

$$g_t(x, y)=0 \tag{7}$$

if x and y belong to different components of Ω_t.

Lemma 5. *At every x and y in Ω_0 with $x\neq y$, the limit*

$$g_0^-(x, y)=\lim_{\varepsilon\downarrow 0}g_{-\varepsilon}(x, y) \tag{8}$$

exists.

Proof. We fix two arbitrary points x and y in Ω_0. Then, there exists some $\delta>0$ such that x and y are contained in $\Omega_{-\delta}$. The sequence of numbers $\{g_{-\varepsilon}(x, y)\}$ forms an increasing sequence when ε decreases to 0. On the other hand, Proposition 2 gives

$$g_{-\varepsilon}(x, y)\leq g_0(x, y). \tag{9}$$

This proves Lemma 5.

Lemma 6. *$g_0^-(x, y)=g_0(x, y)$ for any $x, y\in\Omega_0$ satisfying $x\neq y$.*

Proof. We want to prove that

$$h(x)=g_0(x, y)-g_0^-(x, y) \tag{10}$$

vanishes identically in Ω_0. This function $h(x)$ is the limit of

$$h_\varepsilon(x)=g_0(x, y)-g_{-\varepsilon}(x, y). \tag{11}$$

We have, by Propositions 1 and 2, that

$$0\leq h_\varepsilon(x)\leq g_0(x, y) \qquad \text{if } x\in\Omega_{-\varepsilon}\setminus\{y\}. \tag{12}$$

Since $h_\varepsilon(x)$ is harmonic in $\Omega_{-\delta}$ for any $\delta>\varepsilon>0$, $h(x)$ is harmonic in $\Omega_{-\delta}$ by virtue of Harnack's theorem. This implies that $h(x)$ is harmonic in Ω_0. Therefore, Lemma 6 will be proved if we prove

$$\lim_{x\to\gamma_0}h(x)=0. \tag{13}$$

(12) implies that

$$0\leq\liminf_{x\to\gamma_0}h(x)\leq\limsup_{x\to\gamma_0}h(x)\leq\limsup_{x\to\gamma_0}g_0(x, y)=0. \tag{14}$$

Hence (13) is proved.

Lemma 7. *If x and y are different points in Ω_0,*

$$g_0^+(x, y)=\lim_{\varepsilon\downarrow 0}g_\varepsilon(x, y) \tag{15}$$

exists.

Proof. For any $x\neq y\in\Omega_0$, we have

$$g_\varepsilon(x, y) \geq g_0(x, y) \geq 0. \tag{16}$$

On the other hand $\{g_\varepsilon(x, y)\}$ forms a non-decreasing sequence when ε tends to 0 decreasingly. This proves Lemma 7.

Lemma 8. *For any different points x and y in Ω_0, we have*

$$g_0^+(x, y) = g_0(x, y). \tag{17}$$

Proof. We want to prove

$$h(x) = g_0^+(x, y) - g_0(x, y) \tag{18}$$

vanishes identically in Ω_0. This function is the limit of

$$h_\varepsilon(x) = g_\varepsilon(x, y) - g_0(x, y) \tag{19}$$

when ε goes to 0 decreasingly. The function $h_\varepsilon(x)$ is harmonic in Ω_0 and it satisfies the inequality

$$0 \leq h_\varepsilon(x) = g_\varepsilon(x, y) \leq g_1(x, y) \qquad \text{on } \gamma_0. \tag{20}$$

It follows from this and Harnack's theorem that $h(x)$ is harmonic in Ω_0 and satisfies

$$0 \leq h(x). \tag{21}$$

Lemma 8 will be proved if we prove

$$\lim_{x \to \gamma_0 \setminus \{x^0\}} h(x) = 0, \tag{22}$$

because $h(x)$ is harmonic in Ω_0 and the one point set $\{x^0\}$ is of harmonic measure 0 in γ_0.

(22) follows from (21) and

$$\limsup_{x \to \gamma_0 \setminus \{x^0\}} h(x) \leq 0. \tag{23}$$

Now we prove (23). Assume that x tends to x' in $\gamma_0 \setminus \{x^0\}$. Then, there exists a domain Ω' with the following properties;

(i) $\Omega' \supset \Omega_0$.

(ii) The boundary $\partial\Omega'$ of Ω' is smooth.

(iii) In some neighbourhood U of x', $\partial\Omega'$ coincides with $\gamma_0 \cap U$.

An example of such Ω' is indicated in Fig. 1, where dotted line shows $\partial\Omega_0$.

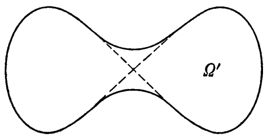

Fig. 1

Let $g'(x, y)$ be the Green function for the Dirichlet problem in Ω'. We claim that

$$g_0^+(x, y) \leq g'(x, y). \tag{24}$$

If this is the case, we can prove

$$\limsup_{x \to x'} h(x) \leq 0. \tag{25}$$

In fact, letting $x \to x'$ in (24), we have

$$\limsup_{x \to x'} h(x) = \limsup_{x \to x'} g_0^+(x, y) \leq \limsup_{x \to x'} g'(x, y) = 0, \tag{26}$$

because of property (iii) of Ω'.

In order to prove (24), we introduce the domain

$$\Omega'_t=\{x\in \boldsymbol{R}^2 \mid \operatorname{dist}(x, \bar{\Omega}')<t\},$$

where dist $(x, \bar{\Omega}')$ denotes the distance from x to $\bar{\Omega}'$. Let $g'_t(x, y)$ denote the Green function for the Dirichlet problem in the domain Ω'_t. Since Ω' is smooth, the Hadamard variational formula implies that

$$\lim_{t\downarrow 0} g'_t(x, y)=g'(x, y) \tag{27}$$

for any $x\neq y$ in Ω'.

For any $t>0$, there exists $\varepsilon_0>0$ such that $\Omega_\varepsilon\subset\Omega'_t$ for any positive ε satisfying $\varepsilon<\varepsilon_0$. This implies that

$$g_\varepsilon(x, y)\leq g'_t(x, y) \qquad \text{for any } \varepsilon\leq\varepsilon_0. \tag{28}$$

Letting $\varepsilon\downarrow 0$, we have

$$g_0^+(x, y)\leq g'_t(x, y) \tag{29}$$

if $x\neq y$ in Ω_0. Letting t tend to 0, we have (24) because of (27). Lemma 8 has been proved.

We have proved

Theorem 2. *For any $x\neq y$ in Ω_0, we have*

$$g_0(x, y)=\lim_{\varepsilon\downarrow 0} g_\varepsilon(x, y) \tag{30}$$

$$g_0(x, y)=\lim_{\varepsilon\uparrow 0} g_\varepsilon(x, y). \tag{31}$$

§3. Proof of Theorem 1. Hadamard variational formula (3) gives

$$g_1(x, y)-g_\varepsilon(x, y)=\int_\varepsilon^1 \frac{d}{dt}g_t(x, y)dt \qquad \text{for } \forall x, y \text{ in } \Omega_0 \tag{32}$$

and

$$g_{-\delta}(x, y)-g_{-1}(x, y)=\int_{-1}^{-\delta} \frac{d}{dt}g_t(x, y)dt \qquad \text{for } \forall x, y\in\Omega_{-1}, \tag{33}$$

if $\varepsilon>0$ and $\delta>0$. Therefore, Theorem 2 implies that

$$\lim_{\varepsilon\downarrow 0}\int_\varepsilon^1 \frac{d}{dt}g_t(x, y)dt \quad \text{and} \quad \lim_{\delta\downarrow 0}\int_{-1}^{-\delta} \frac{d}{dt}g_t(x, y)dt \tag{34}$$

exist. Moreover, we have

$$g_1(x, y)-g_{-1}(x, y)=\lim_{\varepsilon\downarrow 0}\int_\varepsilon^1 \frac{d}{dt}g_t(x, y)dt+\lim_{\delta\downarrow 0}\int_{-1}^{-\delta} \frac{d}{dt}g_t(x, y)dt. \tag{35}$$

The Hadamard variational formula (3) implies that $(d/dt)g_t(x, y)$ is continuous even at $x=y$ if $t\neq 0$. Thus, we put

$$\frac{d}{dt}g_t(y, y)=\lim_{x\to y} \frac{d}{dt}g_t(x, y).$$

Then, we have from (3) that

$$\frac{d}{dt}g_t(y, y)\geq 0 \tag{36}$$

and the Hadamard's inequality (Hadamard [5])

$$\left|\frac{d}{dt}g_t(x, y)\right| \leq \left(\frac{d}{dt}g_t(x, x)\right)^{1/2}\left(\frac{d}{dt}g_t(y, y)\right)^{1/2}. \tag{37}$$

Since $h_\varepsilon(x)=g_1(x, y)-g_\varepsilon(x, y)$ is harmonic even at $x=y$, we can take the limit of both sides of (32) when x tends to y. And we have

$$h_\varepsilon(y)=\int_\varepsilon^1 \frac{d}{dt}g_t(y, y)dt, \qquad \text{for any } y \in \Omega_0. \tag{38}$$

The harmonic function $h_\varepsilon(x)$ satisfies

$$0 \leq h_\varepsilon(x) \leq g_1(x, y) \tag{39}$$

for any x in Ω_0 and it converges to the harmonic function $g_1(x, y)-g_0(x, y)$ if $x \neq y$. Therefore $h_\varepsilon(x)$ converges also at $x=y$ by virtue of mean value theorem and Harnack's theorem. Hence we have

$$\lim_{\varepsilon \downarrow 0} h_\varepsilon(y)=\lim_{\varepsilon \downarrow 0}\int_\varepsilon^1 \frac{d}{dt}g_t(y, y)dt. \tag{40}$$

This and (36) imply that $(d/dt)g_t(y, y)$ is Lebesgue integrable over [0, 1], that is,

$$\int_0^1 \frac{d}{dt}g_t(y, y)dt < \infty \qquad \text{for any } y \text{ in } \Omega_0. \tag{41}$$

Similarly, we can prove

$$\int_{-1}^0 \frac{d}{dt}g_t(y, y)dt < \infty \qquad \text{for any } y \text{ in } \Omega_0. \tag{42}$$

Thus, we have proved

$$\int_{-1}^1 \frac{d}{dt}g_t(y, y)dt < \infty \qquad \text{for any } y \text{ in } \Omega_0. \tag{43}$$

This and Hardamard's inequality (37) prove

$$\int_{-1}^1 \left|\frac{d}{dt}g_t(x, y)\right| dt < \infty. \tag{44}$$

It follows from this and (35) that

$$g_1(x, y)-g_{-1}(x, y)=\int_{-1}^1 \frac{d}{dt}g_t(x, y)dt$$

as a Lebesgue integral. Theorem 1 has been proved.

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