40. Probabilistic Construction of the Solution of Some Higher Order Parabolic Differential Equation

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§ 1. **Introduction.** The purpose of this note is to give a probabilistic solution of higher order partial differential equations of some specific type. We first recall that the solution of the heat equation

(1) $\dfrac{\partial u}{\partial t}=\dfrac{1}{2}\left(\dfrac{\partial}{\partial x}\right)^2 u,\ (t, x)\in(0,\infty)\times \boldsymbol{R}^1$

(2) $u(0,x)=u_0(x)$

is expressed in terms of a Brownian motion $\{\boldsymbol{B}_t\}_{t\geqq 0}$ in the form

(3) $u(t,x)=\boldsymbol{E}[u_0(x+\boldsymbol{B}_t)]$,

where $\boldsymbol{E}$ means the expectation. The key formula to prove that (3) satisfies (1) is $(d\boldsymbol{B}_t)^2=dt$. Being inspired by this formula, we take another Brownian motion $\boldsymbol{w}_t$ to expect that a formal formula $(d\boldsymbol{B}_{w_t})^4=dt$ would hold, so that the process $\boldsymbol{B}_{w_t}$ is related to the operator $\left(\dfrac{\partial}{\partial x}\right)^4$. More precisely, our problem is to express the solution of equation

(4) $\dfrac{\partial u}{\partial t}=\left(\dfrac{\partial}{\partial x}\right)^4 u$

in a similar form to (3) by using the process $\boldsymbol{B}_{w_t}$. However, $\boldsymbol{B}_{w_t}$ can not be viewed as a motion of some particle since the Brownian motion $\boldsymbol{B}_t$ as a diffusion process with generator $\dfrac{1}{2}\left(\dfrac{\partial}{\partial x}\right)^2$ is defined only for $t\geqq 0$, while $\boldsymbol{w}_t$ can take negative values. To overcome this difficulty, we need some trick as is illustrated in what follows.

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§ 2. **Simple case.** First we discuss a simple equation

(5) $\dfrac{\partial u}{\partial t}=\dfrac{1}{8}\left(\dfrac{\partial}{\partial x}\right)^4 u,\ (t, x)\in(0,\infty)\times \boldsymbol{R}^1.$

Let $\{\bar{\boldsymbol{B}}_t\}_{t\in \boldsymbol{R}^1}$ be a complex-valued stochastic process given by

$$\bar{\boldsymbol{B}}_t=\begin{cases}\boldsymbol{B}_t, & t\geqq 0,\\ i\boldsymbol{B}_{-t}, & t\leqq 0.\end{cases}$$

Denote by $\mathscr{D}_1$ the class of real-valued functions $f(x)$ defined on $\boldsymbol{R}^1$ which are extensible to entire functions $\bar{f}(z)$ on $\boldsymbol{C}^1$ satisfying the following conditions (6) and (7).

(6) $\bar{f}(x)=f(x)$ for $x\in \boldsymbol{R}^1$.

(7) $|\bar{f}(z)|\exp\{-h|z|^2\}$, $\left|\dfrac{\partial \bar{f}}{\partial x}(z)\right|\exp\{-h|z|^2\}$ and

$\left|\dfrac{\partial \bar{f}}{\partial y}(z)\right|\exp\{-h|z|^2\}$ are bounded on $\boldsymbol{C}^1$ for each $h>0$, where $z=x+iy$.

The heat equation is, in general, not well-posed for the past, as is well-known. However, we have the following theorem by using a probabilistic method.

Theorem 1. *For any* $u_0\in\mathcal{D}_1$, *the function* $v(t,x)=\boldsymbol{E}[\bar{u}_0(x+\bar{\boldsymbol{B}}_t)]$ *satisfies the heat equation* (1) *for* $(t,x)\in(\boldsymbol{R}^1-\{0\})\times\boldsymbol{R}^1$ *and the initial condition* (2).

Outline of the proof. It is obvious that the function v satisfies (1) for $(t,x)\in(0,\infty)\times\boldsymbol{R}^1$ and the initial condition (2). Set $v(t,z)=\boldsymbol{E}[\bar{u}_0(z+\bar{\boldsymbol{B}}_t)]$, $z=x+iy\in\boldsymbol{C}^1$. For $t<0$, since $v(t,z)=\boldsymbol{E}[\bar{u}_0(z+i\boldsymbol{B}_{-t})]$,

$$\begin{aligned}\frac{\partial v}{\partial t}(t,z)&=-\frac{\partial}{\partial s}\boldsymbol{E}[\bar{u}_0(x+i(y+\boldsymbol{B}_s))]\big|_{s=-t}\\&=-\frac{1}{2}\left(\frac{\partial}{\partial y}\right)^2\boldsymbol{E}[\bar{u}_0(x+i(y+\boldsymbol{B}_{-t}))]\\&=\frac{1}{2}\left(\frac{\partial}{\partial x}\right)^2\boldsymbol{E}[\bar{u}_0(x+i(y+\boldsymbol{B}_{-t}))]\\&=\frac{1}{2}\left(\frac{\partial}{\partial x}\right)^2 v(t,z).\end{aligned}$$

We have the conclusion by taking $y=0$ in this equality.

Let $\boldsymbol{w}_t$ be a 1-dimensional Brownian motion independent of $\boldsymbol{B}_t$. Our main result can be summarized in the following theorem.

Theorem 2. *For any* $u_0\in\mathcal{D}_1$, *the function* $u(t,x)=\boldsymbol{E}[\bar{u}_0(x+\bar{\boldsymbol{B}}_{\boldsymbol{w}_t})]$ *is a solution of* (5) *with the initial condition* (2).

Outline of the proof. It is obvious that the function u satisfies the initial condition (2). Set $u(t,s,x)=\boldsymbol{E}[v(s+\boldsymbol{w}_t,x)]$, $(t,s,x)\in[0,\infty)\times\boldsymbol{R}^1\times\boldsymbol{R}^1$. Then, by Theorem 1, $\dfrac{\partial u}{\partial s}=\dfrac{1}{2}\left(\dfrac{\partial}{\partial x}\right)^2 u$ holds for $(s,x)\in\boldsymbol{R}^1\times\boldsymbol{R}^1$. Since $\dfrac{\partial u}{\partial t}=\dfrac{1}{2}\left(\dfrac{\partial}{\partial s}\right)^2 u$ holds for $(t,s)\in[0,\infty)\times\boldsymbol{R}^1$,

$$\frac{\partial u}{\partial t}=\frac{1}{2}\left(\frac{1}{2}\left(\frac{\partial}{\partial x}\right)^2\right)^2 u=\frac{1}{8}\left(\frac{\partial}{\partial x}\right)^4 u.$$

We have the conclusion by taking $s=0$ in this equality.

§3. Generalization. Our method is also applicable to a more general equation

(8) $\dfrac{\partial u}{\partial t}=\boldsymbol{P}(A)u$, $(t,x)\in(0,\infty)\times\boldsymbol{R}^d$,

where $\boldsymbol{P}$ is a polynomial of degree 2 such as

(9) $P(a)=pa^2+qa,\ p\geqq 0,\ q\in \boldsymbol{R}^1$,

and where A is a uniformly elliptic differential operator of the form

(10) $$A=\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}+\sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}$$

with bounded continuous coefficients $a_{ij}(x)$ and $b_i(x)$. Associated with A is a d-dimensional diffusion process $X_t(x)$ with generator A and with starting point $x\in \boldsymbol{R}^d$. Denote by $\mathcal{D}_2$ the class of real-valued functions $f(x)$ defined on $\boldsymbol{R}^d$ which are extensible to functions $\bar{f}(x, y)$ defined on $\boldsymbol{R}^d\times \boldsymbol{R}^n$ for some n satisfying the following conditions (11)–(13).

(11) $\bar{f}(x, 0)=f(x),\ x\in \boldsymbol{R}^d$.

(12) There exists a uniformly elliptic differential operator $\tilde{A}$ of the same type as in (10) defined on $\boldsymbol{R}^n$ such that $(A_x+\tilde{A}_y)\bar{f}(x, y)=0$.

(13) $|\bar{f}(x, y)| \exp\{-h(|x|^2+|y|^2)\}$, $\left|\dfrac{\partial \bar{f}}{\partial x_i}(x, y)\right| \exp\{-h(|x|^2+|y|^2)\}$, $i=1, \cdots d$, and $\left|\dfrac{\partial \bar{f}}{\partial y_j}(x, y)\right| \exp\{-h(|x|^2+|y|^2)\}$, $j=1, \cdots, n$, are bounded on $\boldsymbol{R}^d\times \boldsymbol{R}^n$ for each $h>0$.

Denote by $\tilde{X}_t$ an n-dimensional diffusion process with the generator $\tilde{A}$ starting from $0\in \boldsymbol{R}^n$, and define a $(d+n)$-dimensional stochastic process $\{\bar{X}_t(x)\}_{t\in \boldsymbol{R}^1, x\in \boldsymbol{R}^d}$ by

$$\bar{X}_t(x)=\begin{cases}(X_t(x), 0), & t\geqq 0,\\ (x, \tilde{X}_{-t}), & t\leqq 0.\end{cases}$$

Let w_t be a 1-dimensional Brownian motion independent of $\bar{X}_t(x)$, and set $Y_t=\sqrt{2p}\,w_t+qt$. Then, we have the following theorem as a generalization of Theorem 2.

Theorem 3. *For any $u_0\in \mathcal{D}_2$, the function $u(t, x)=E[\bar{u}_0(\bar{X}_{Y_t}(x))]$ is a solution of* (8) *with the initial condition* (2).

This theorem can be proved in a similar way to Theorem 2.

Repeating these procedures to replace the time variable t with independent Brownian motions, we can construct a solution of (8) for a polynomial $P(a)$ of degree 2^m expressed in the form

(14) $P(a)=P_1\circ P_2\circ \cdots \circ P_m(a),\ P_i(a)=p_i a^2+q_i a\ (i=1, 2, \cdots, m)$.

§4. Remarks. Finally, we note that our method is in line with Bochner's subordination. Bochner's subordination reads as follows ([1], [3]). Let $\{T_t\}_{t\geqq 0}$ be a semi-group defined on some Banach space, and let $\{F_t(dx)\}_{t\geqq 0}$ be a family of infinitely divisible probability distributions supported by $[0, \infty)$ such that there is a function ψ satisfying

(15) $\displaystyle\int_0^\infty e^{-\lambda x}F_t(dx)=e^{-t\psi(\lambda)}$ for every t.

If we define $\{T_t^\psi\}_{t\geqq 0}$ by

(16) $\displaystyle T_t^\psi=\int_0^\infty T_s F_t(ds)$,

then $\{T_t^{\psi}\}_{t \geqq 0}$ is again a semi-group and the generator is $-\psi(-A)$, where A is the generator of $\{T_t\}_{t \geqq 0}$.

In our case where the support of the subordinator $\{Y_t\}_{t \geqq 0}$ is all over R^1, we must construct $\{T_t\}_{t \in R^1}$ as a group instead of a semi-group such as in Theorem 1. The advantage of our method, although the class of initial functions is smaller than analytic methods, is to obtain the visualized expression of the solution (Theorems 2 and 3).

These results will be published elsewhere with a detailed proof.

Note. K. Hochberg [2] discusses the same type of equations as ours by a different probabilistic method by using a signed measure over a path space.

References

[1] S. Bochner: Harmonic Analysis and the Theory of Probability. Univ. of California Press (1955).

[2] K. Hochberg: A signed measure on path space related to Wiener measure. Ann. Prob., **6**, 433–458 (1978).

[3] E. Nelson: A functional calculus using singular Laplace integrals. Trans. Amer. Math. Soc., **88**, 400–413 (1958).