39. Perturbation of Domains and Green Kernels of Heat Equations. II

By Shin OZAWA
Department of Mathematics, University of Tokyo

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§. **Introduction.** In our previous paper [2], the author gave Hadamard's variational formula of the Green kernels of heat equations with the Dirichlet boundary condition and gave the variational formula of the trace $T_r(t)$ of them. See also [3]. In this paper, in § 1, we study Hadamard's variational formula of the Green kernels of heat equations with the Neumann and the third boundary condition. In Theorem 2, we shall also give the variational formula of the trace $T_r(t)$ associated with them. In relation to Theorems 2 and 3, we study the dependence on domains of the eigenvalue of the Laplacian with the Neumann and the third boundary condition. In § 2, we give a rough sketch of the proof of Theorem 1. Details of the proof of Theorems 1 and 2 will be given elsewhere.

§ 1. **Hadamard's variational formulas.** Let Ω be a bounded domain in $\mathbf{R}^n$ with C^∞ boundary γ. Let $\rho(x)$ be a smooth function on γ and ν_x be the exterior unit normal vector at $x \in \gamma$. For sufficiently small $\varepsilon \geq 0$, let Ω_ε be the bounded domain whose boundary γ_ε is defined by

$$\gamma_\varepsilon = \{x + \varepsilon\rho(x)\nu_x \,;\, x \in \gamma\}.$$

Let $U_\varepsilon(x, y, t)$ be the Green kernel of heat equation with the third boundary condition, that is, $U_\varepsilon(x, y, t)$ has the following properties:

$$(1.1)_k \qquad \begin{cases} \left(\dfrac{\partial}{\partial t} - \Delta_x\right) U_\varepsilon(x, y, t) = 0, & x, y \in \Omega_\varepsilon,\ t > 0 \\ \lim\limits_{t \to +0} U_\varepsilon(x, y, t) = \delta(x-y), & x, y \in \Omega_\varepsilon \\ \left(\dfrac{\partial}{\partial \nu_x^\varepsilon} + k\right) U_\varepsilon(x, y, t) = 0, & x \in \gamma_\varepsilon,\ y \in \Omega_\varepsilon,\ t > 0, \end{cases}$$

where k is a fixed non-negative constant and $\dfrac{\partial}{\partial \nu_x^\varepsilon}$ denotes the derivative along the exterior normal direction at $x \in \gamma_\varepsilon$. We abbreviate $U_0(x, y, t)$ as $U(x, y, t)$. We give a notation. We fix $z \in \gamma$ and take an orthonormal basis $(z_1, \cdots, z_{n-1})$ on the tangent hyperplane at z. Then, we put

$$\langle \nabla_\gamma a(z), \nabla_\gamma b(z) \rangle = \sum_{j=1}^{n-1} \frac{\partial a}{\partial z_j}(z) \frac{\partial b}{\partial z_j}(z)$$

for $a(z), b(z) \in C^\infty(\gamma)$.

Put

$$\delta U(x, y, t)=\lim_{\varepsilon\to 0} \varepsilon^{-1}(U_\varepsilon(x, y, t)-U(x, y, t)),$$

for $x, y \in \Omega$, $t>0$, then we have the following

Theorem 1.

$$\begin{aligned}\delta U(x, y, t)= & -\int_0^t d\tau \int_\gamma \langle \nabla_\gamma U(x, z, t-\tau), \nabla_\gamma U(y, z, \tau)\rangle \rho(z) d\sigma_z \\ & -\frac{\partial}{\partial t}\int_0^t d\tau \int_\gamma U(x, z, t-\tau) U(y, z, \tau)\rho(z) d\sigma_z \\ & +\int_0^t d\tau \int_\gamma U(x, z, t-\tau) U(y, z, \tau)(k^2-(n-1)kH_1(z))\rho(z) d\sigma_z,\end{aligned}$$

for any fixed $x, y \in \Omega$, $t>0$. Here $H_1(z)$ denotes the first mean curvature of γ at z, and $d\sigma_z$ denotes the surface element of γ.

Let $T_r(t\,;\varepsilon)$ denote the trace of $U_\varepsilon(x, y, t)$ on Ω_ε which is defined by

$$T_r(t\,;\varepsilon)=\int_{\Omega_\varepsilon} U_\varepsilon(x, x, t)dx.$$

Then, we have the following

Theorem 2. *For any fixed $t>0$,*

$$\delta T_r(t)=\tilde{\int}_\Omega \delta U(x, x, t)dx+\int_\gamma U(x, x, t)\rho(x)d\sigma_x,$$

where

$$\delta T_r(t)=\lim_{\varepsilon\to 0} \varepsilon^{-1}(T_r(t\,;\varepsilon)-T_r(t\,;0)).$$

Here

$$\tilde{\int}_\Omega \delta U(x, x, t)dx$$

denotes the distribution on $(0, \infty)$ which is defined by the mapping

$$C_0^\infty(0, \infty) \ni \Phi(t) \mapsto \int_\Omega dx \int_0^\infty \Phi(t)\delta U(x, x, t)dt.$$

Let $0\geq\lambda_1\geq\lambda_2\geq\lambda_3\geq\cdots$ be the eigenvalues of the following problem:

$$(1.2)_k \qquad \begin{cases}\Delta u(x)=\lambda u(x) & \text{in } \Omega \\ \left(\dfrac{\partial}{\partial\nu_x}+k\right)u(x)=0, & x\in\gamma.\end{cases}$$

Here $k\geq 0$. Let $\{\varphi_j(x)\}_{j=1}^\infty$ be a complete orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions of $(1.2)_k$. We assume that $\varphi_j(x)$ belongs to the eigenspace associated with λ_j.

It is well known that

$$U(x, y, t)=\sum_{j=1}^\infty e^{\lambda_j t}\varphi_j(x)\varphi_j(y).$$

Therefore, by Theorems 1 and 2, we have the following

Theorem 3.

$$\delta T_r(t)=t\sum_{j=1}^\infty e^{\lambda_j t}\int_\gamma Q_j(z)\rho(z)d\sigma_z,$$

where

$$Q_j(z) = -|\nabla_\gamma \varphi_j(z)|^2 + (k^2 - (n-1)kH_1(z) - \lambda_j)\varphi_j(z)^2.$$

Here

$$|\nabla_\gamma \varphi_j(z)|^2 = \langle \nabla_\gamma \varphi_j(z), \nabla_\gamma \varphi_j(z) \rangle.$$

The following theorem is well known.

Theorem 4 (Courant-Hilbert [1]). *Let Ω be a bounded domain in $\mathbf{R}^2$ and $\lambda_j^*(\Omega)$ denotes the j-th eigenvalue of $-\Delta$ with the Dirichlet boundary condition, then*

$$\lambda_j^*(\Omega) \le \lambda_j^*(\omega)$$

for any subdomain ω.

Theorem 4 states that the eigenvalues of the Laplacian with the Dirichlet boundary condition are monotonously dependent on domains. In Uchiyama [4], he showed that the eigenvalues of the Laplacian are not monotonously dependent on domains in the case of the Neumann boundary condition. It should be remarked that $Q_j(z)$ may change its sign in the case of the Neumann boundary condition. It must be a reason why such a non-monotonous dependence on domains of eigenvalues occurrs. Recently the present author was noticed by Prof. J. Watanabe and Mr. S. Kaizu that they obtained the variational formulas of the eigenvalues of the Laplacian with the Neumann and the third boundary condition and that they studied the dependence of eigenvalues on domains. They do not use the fundamental solution of the heat equation for their studies. Our formula in Theorem 3 leads to their formula when the eigenvalues of $(1.2)_k$ are all simple eigenvalues.

§ 2. Sketch of the proof of Theorem 1. We shall give a rough sketch of the proof of Theorem 1. For fixed $x \in \Omega$, let $\tilde{U}(x, y, t)$, $y \in \mathbf{R}^n$, $t > 0$, be Whitney's extension of $U(x, y, t)$ with the following properties;

$$\lim_{t \to +0} \tilde{U}(x, y, t) = 0 \qquad \text{for } y \notin \bar{\Omega}.$$

Then, for fixed $x, y \in \Omega$ and $t > 0$, we have

$$\begin{aligned} &U_\varepsilon(x, y, t) - U(x, y, t) \\ &\quad = -\int_0^t d\tau \left(\frac{\partial}{\partial \tau} \int_{\Omega_\varepsilon} U(x, z, t-\tau) \tilde{U}(y, z, \tau) dz \right) \\ &\quad = \int_0^t d\tau \int_{\Omega_\varepsilon} (\tilde{U}(y, z, \tau) \Delta_z U_\varepsilon(x, z, t-\tau) \\ &\qquad\qquad - \Delta_z \tilde{U}(y, z, \tau) U_\varepsilon(x, z, t-\tau)) dz + 0(\varepsilon^2). \end{aligned}$$

Here we have used the following inequality; For z in a sufficiently small fixed neighbourhood of γ,

$$\left| \left(\frac{\partial}{\partial t} - \Delta_z \right) \tilde{U}(y, z, t) \right| \le C \operatorname{dist}(z, \gamma)^2,$$

where C is a constant independent of t. Then, by Green's identity and by the boundary condition in $(1.1)_k$, we have

$$
\begin{aligned}
&U_\varepsilon(x, y, t) - U(x, y, t) \\
(2.1) \qquad &= -\int_0^t d\tau \int_{\gamma_\varepsilon} U_\varepsilon(x, z, t-\tau)\left(\frac{\partial \tilde{U}}{\partial \nu_z^\varepsilon}(y, z, \tau) + k\tilde{U}(y, z, \tau)\right) d\sigma_z^\varepsilon \\
&\quad + 0(\varepsilon^2),
\end{aligned}
$$

where $d\sigma_z^\varepsilon$ denotes the surface element of γ_ε. For any $z \in \gamma_\varepsilon$, there is the unique point $w \in \gamma$ such that $z = w + \varepsilon\rho(w)\nu_w$. By simple geometrical observation, we have

$$
\begin{aligned}
&\frac{\partial \tilde{U}}{\partial \nu_z^\varepsilon}(y, z, \tau) + k\tilde{U}(y, z, \tau)\Big|_{z = w + \varepsilon\rho(w)\nu_w} \\
(2.2) \qquad &= \left(\frac{\partial U}{\partial \nu_w}(y, w, \tau) + kU(y, w, \tau)\right) \\
&\quad + \varepsilon\left(\frac{\partial^2 U}{\partial \nu_w^2}(y, w, \tau) + k\frac{\partial U}{\partial \nu_w}(y, w, \tau)\right)\rho(w) \\
&\quad - \varepsilon\langle \nabla_\gamma \rho(w), \nabla_\gamma U(y, w, \tau)\rangle + 0(\varepsilon^2).
\end{aligned}
$$

Such a formula is given in [3]. By using *a priori* estimates of Schauder, we can examine the ε-dependence of the fundamental solution $U_\varepsilon(x, y, t)$. Together with (2.1) and (2.2), we obtain

$$
\begin{aligned}
\delta U(x, y, t) &= -\int_0^t d\tau \int_\gamma U(x, z, t-\tau)\frac{\partial^2 U}{\partial \nu_z^2}(y, z, \tau)\rho(z) d\sigma_z \\
(2.3) \qquad &\quad + k^2 \int_0^t d\tau \int_\gamma U(x, z, t-\tau)U(y, z, \tau)\rho(z) d\sigma_z \\
&\quad + \int_0^t d\tau \int_\gamma \langle \nabla_\gamma \rho(z), \nabla_\gamma U(y, z, \tau)\rangle d\sigma_z.
\end{aligned}
$$

On the other hand, for $z \in \gamma$, we have

$$
(2.4) \qquad \left(\frac{\partial^2}{\partial \nu_z^2} + \nabla_\gamma^2 + (n-1)H_1(z)\frac{\partial}{\partial \nu_z} - \frac{\partial}{\partial \tau}\right) U(x, z, \tau) = 0,
$$

where ∇_γ^2 denotes the Laplacian on the submanifold γ of $\mathbf{R}^n$ with the induced metric from $\mathbf{R}^n$. By (2.3) and (2.4), we get Theorem 1.

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Erratum. In Theorem 7 in [2, p. 325], the term $(n!)^{-1}C_n\lambda^{1+n/2}$ should be read $\Gamma\left(2 + \frac{n}{2}\right)^{-1} C_n \lambda^{1+(n/2)}$.

References

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