38. *Note on Certain Nonlinear Evolution Equations of Second Order*

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1. Introduction. In this note we consider nonlinear evolution equations of the form

(1.1) $$u''(t)+Au(t)+B(t)u'(t)=f(t), \qquad 0\leqq t\leqq T,$$

with initial conditions

(1.2) $$u(0)=u_0 \quad \text{and} \quad u'(0)=u_1,$$

($u'(t)=du(t)/dt$, $u''(t)=d^2u(t)/dt^2$), where A is a nonlinear operator and each $B(t)$ is a formally self-adjoint positive operator.

When $B(t)\equiv 0$, there are a great number of results on non-existence of global weak solutions of (1.1) (see e.g. Knops-Straughan [4] and the cited papers therein). However, as for the existence of a global weak solution for an abstract Cauchy problems (1.1) and (1.2), where A is a genuinely nonlinear operator, it seems that there are few results except for Tsutsumi's [8]. He obtained sufficient conditions for the global existince under the presence of the dissipative term $B(t)u'(t)$.

The purpose of the present note is to show the existence of a global weak solution of (1.1) and (1.2) satisfying a certain inequality of energy type. Especially, we intend to weaken the assumptions of Tsutsumi [8] so that the result can be applied to a wider class of nonlinear partial differential equations.

2. Assumptions and result. Let H be a real separable Hilbert space with inner product $(\cdot,\cdot)$ and norm $|\cdot|_H$. Let W be a second real separable Hilbert space with norm $|\cdot|_W$ and let V be a real separable reflexive Banach space with norm $|\cdot|_V$. Suppose that

$$V\subset W\subset H,$$

where each injection is dense and continuous. Furthermore, the injection of W into H is compact. As usual, we identify H with its own dual and denote by V^* and W^* the dual spaces of V and W, respectively. Then the following inclusion relation holds:

$$V\subset W\subset H\subset W^*\subset V^*.$$

The pairing between $x^*\in V^*$ (resp. $x^*\in W^*$) and $x\in V$ (resp. $x\in W$) is simply denoted by (x^*,x); if $x,x^*\in H$, this is the ordinary inner product in H.

Throughout this note we put the following assumptions on the nonlinear operator $A: V\to V^*$.

(A. 1) For each $u \in V$, $Au \in V^*$ is the Gâteaux differential of a convex functional F_A at u, which is lower semicontinuous on V.

(A. 2) For each $c>0$, $\{u \in V; F_A(u) \leqq c\}$ is bounded in V.

(A. 3) A maps every bounded set of V into a bounded set of V^*.

For the linear operator $B(t): W \to W^*$, we assume the following.

(B. 1) For each $t \in [0, T]$, $B(t)$ is a linear operator associated with a symmetric bilinear form $b(t; \cdot, \cdot)$ on W, which satisfies

$$|b(t; u, v)| \leqq b_1 |u|_W |v|_W \quad \text{and} \quad b(t; u, u) \geqq b_2 |u|_W^2 \quad \text{for } {}^\forall u, v \in W,$$

where b_1 and b_2 are some positive constants independent of t.

(B. 2) For each $u, v \in W$, $t \mapsto b(t; u, v)$ is continuously differentiable on $[0, T]$ and $\dot{b}(t; u, v)$ $(\equiv db(t; u, v)/dt)$ has the following property: If $u_n \to u$ weakly in W as $n \to \infty$, then $\limsup_{n\to\infty} \dot{b}(t; u_n, u_n) \leqq \dot{b}(t; u, u)$ for every $t \in [0, T]$.

Under these assumptions we have the main result.

Theorem 2.1. *Let $u_0 \in V$, $u_1 \in H$ and $f \in L^2(0, T; H)$. Then there exists a function u such that*

(2.1) $$u \in L^\infty(0, T; V),$$

(2.2) $$u' \in L^\infty(0, T; H) \cap L^2(0, T; W),$$

(2.3) $$u'' \in L^2(0, T; V^*),$$

and satisfies (1.1), (1.2) *and the following inequality; for any positive function $\psi \in C^1[0, T]$*

(2.4) $$\psi(t)E(u(t)) + \int_s^t \psi(r) b(r; u'(r), u'(r))dr \leqq \psi(s)E(u(s)) + \int_s^t \psi'(r)E(u(r))dr + \int_s^t \psi(r)(f(r), u'(r))dr, \quad a.e.\ 0 \leqq s \leqq t \leqq T,$$

where

$$E(u(t)) = \frac{1}{2}|u'(t)|_H^2 + F_A(u(t)).$$

Remark 2.2. If the injection of V into W is also compact, then the conclusion of Theorem 2.1 holds true with (B. 2) replaced by the following weaker assumption:

(B. 2)$'$ For each $u, v \in W$, $t \mapsto b(t; u, v)$ is continuous on $[0, T]$.

Remark 2.3. Our assumptions (A. 1)–(A. 3) generalize the corresponding ones of Tsutsumi [8]; in particular, it is unnecessary to assume the homogeneity condition of A.

3. Outline of the proof. First we shall prepare some lemmas to prove Theorem 2.1.

Lemma 3.1. *A is a maximal monotone and demicontinuous operator from V to V^*.*

Proof. By (A. 1), it is easily shown that A is a maximal monotone operator from V to V^* (see e.g. Barbu [1, Chap. 2, § 2]). So the demicontinuity of A follows from the result of Rockafellar [7, Cor. 1.1].

Lemma 3.2. *Let $u \in C^1([0,T]; V)$. Then*

$$\frac{d}{dt}F_A(u(t)) = \left(Au(t), \frac{d}{dt}u(t)\right) \qquad \textit{for every } t \in [0,T].$$

Proof. By the definition of the subdifferential (see e.g. [1]),

$$(Au(t+h), u(t+h)-u(t)) \geqq F_A(u(t+h)) - F_A(u(t)) \geqq (Au(t), u(t+h)-u(t)). \tag{3.1}$$

Dividing (3.1) by h and letting $h \to 0$, we obtain the conclusion. (Note that $t \mapsto Au(t)$ is weakly continuous in V^* by Lemma 3.1.)

Let $1 \leqq p \leqq \infty$. We recall the fact that, for any $u \in L^p(0,T; V)$, $t \mapsto Au(t)$ is strongly measurable in V^* by Lemma 3.1 and the result of Brezis [2, Appendice IV]. Define the operator $\mathcal{A}: L^p(0,T;V) \to L^{p'}(0,T;V^*)$ $(1/p+1/p'=1)$ with the domain $D(\mathcal{A})$ as follows:

$$D(\mathcal{A}) = \{u \in L^p(0,T;V);\, Au \in L^{p'}(0,T;V^*)\}$$
$$(\mathcal{A}u)(t) = Au(t) \qquad \text{for a.e. } t \in [0,T].$$

Then we have the following lemma whose proof can be found in [2, Appendice I].

Lemma 3.3. *$\mathcal{A}$ is a maximal monotone operator from $L^p(0,T;V)$ to $L^{p'}(0,T;V^*)$.*

Now we shall begin the proof of Theorem 2.1. It is very standard, so we only sketch it here. For details, see Tsutsumi [8].

We employ the Galerkin's method and take $\{w_j\}$ as the basis. Define approximate functions $u_m(t)$ as follows;

$$u_m(t) = \sum_{j=1}^{m} a_{jm}(t) w_j,$$

where unknown functions a_{jm} are determined by the following ordinary differential equations

$$(u_m''(t), w_j) + (Au_m(t), w_j) + b(t; u_m'(t), w_j) = (f(t), w_j), \qquad j=1,2,\cdots,m,$$

with initial conditions

$$u_m(0) = u_{0,m}, \qquad u_{0,m} = \sum_{j=1}^{m} \alpha_{jm} w_j \to u_0 \quad \text{strongly in } V \text{ as } m \to \infty,$$

$$u_m'(0) = u_{1,m}, \qquad u_{1,m} = \sum_{j=1}^{m} \beta_{jm} w_j \to u_1 \quad \text{strongly in } H \text{ as } m \to \infty.$$

Having proved Lemmas 3.2 and 3.3, we can repeat the same procedure as in [8] with an obvious modification. We can, therefore, extract a subsequence $\{u_\mu\}$ of $\{u_m\}$, which converges (in the sense of [8]) to a weak solution u of (1.1) and (1.2) satisfying (2.1)–(2.3). Note that the convergence properties of $\{u_\mu\}$ in [8, (2.14)–(2.23)] still remain true.

To prove that the weak solution u satisfies (2.4), we use the following lemma which is obtained as a consequence of the above proof (cf. [9, 4.2]).

Lemma 3.4. *For any function $\phi \in C[0, T]$ and $t \in [0, T]$,*

$$\lim_{\mu\to\infty}\int_0^t \phi(s)(Au_\mu(s), u_\mu(s))ds=\int_0^t \phi(s)(Au(s), u(s))ds. \tag{3.2}$$

Now we note that the following inequality holds by (A. 1):

$$F_A(u(t))-F_A(u_\mu(t)) \geqq (Au_\mu(t), u(t)-u_\mu(t)), \qquad {}^\forall t \in [0, T]. \tag{3.3}$$

Hence, with the help of (3.3) and the lower semicontinuity of F_A, (3.3) leads to the following: For any function $\phi \in C[0, T]$ and $t \in [0, T]$,

$$\lim_{\mu\to\infty}\int_0^t \phi(s)F_A(u_\mu(s))ds=\int_0^t \phi(s)F_A(u(s))ds, \tag{3.4}$$

which, in particular, implies that

$$\liminf_{\mu\to\infty} F_A(u_\mu(t))=F_A(u(t)) \qquad \text{for a.e. } t \in [0, T]. \tag{3.5}$$

Recall that the equality in (2.4) holds true for $0 \leqq s \leqq t \leqq T$ if u is replaced by u_μ (use Lemma 3.2). Hence taking the inferior limit of the both sides of the resulting expression and using the convergence properties (3.4), (3.5) and [8, (2.15), (2.23)], we see that u satisfies (2.4).

4. Applications. Let Ω be a bounded domain in R^n_x with smooth boundary Γ. We consider the following two examples.

Example 4.1. Let $J(\xi)$ be a convex $C^1(R^n_\xi)$-function satisfying

$$\alpha_1|\xi|^p \leqq J(\xi) \leqq \alpha_2(|\xi|^p+1), \qquad {}^\forall \xi \in R^n_\xi,$$
$$|\partial J(\xi)/\partial\xi_i| \leqq \alpha_3(|\xi|^{p-1}+1), \qquad {}^\forall \xi \in R^n_\xi,\ i=1, 2, \cdots, n,$$

with

$$p \geqq 2,\ \alpha_1, \alpha_2, \alpha_3>0 \quad \text{and} \quad |\xi|^2=\sum_{i=1}^n \xi_i^2.$$

Set

$$a_i(\xi)=\partial J(\xi)/\partial\xi_i, \qquad i=1, 2, \cdots, n.$$

We consider the following initial boundary value problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}-\displaystyle\sum_{i=1}^n \frac{\partial}{\partial x_i}(a_i(\text{grad } u))-b(t)\Delta\frac{\partial u}{\partial t}=f & \text{in } \Omega\times[0, T], \\ u(x, t)=0 & \text{on } \Gamma\times[0, T], \\ u(x, 0)=u_0(x), \quad \dfrac{\partial u}{\partial t}(x, 0)=u_1(x) & \text{in } \Omega, \end{cases}$$

where f, u_0 and u_1 are given functions and $b \in C^1[0, T]$ is a monotone non-increasing positive function (cf. Tsutsumi [8]).

Take $H=L^2(\Omega)$, $V=W_0^{1,p}(\Omega)$ and $W=H_0^1(\Omega)$. If we put

$$F_A(u)=\int_\Omega J(\text{grad } u(x))dx,$$

we easily see that our hypotheses are satisfied. Thus we can apply Theorem 2.1 to the above problem.

Example 4.2. Next we consider nonlinear partial integro-differential equations of the form

$$\frac{\partial^2 u}{\partial t^2}-M\Big(\int_\Omega |\text{grad } u(x)|^2\, dx\Big)\Delta u-b(t)\Delta\frac{\partial u}{\partial t}=f \qquad \text{in } \Omega\times[0, T],$$

with the same initial and boundary conditions as Example 4.1 (cf.

Dickey [3], Medeiros [5] and Pohozaev [6]). Suppose that M is a continuous and monotone non-decreasing function on $[0, \infty)$ satisfying

$$M(0) \geqq 0 \quad \text{and} \quad \int_0^\infty M(r)dr = \infty.$$

Take $H = L^2(\Omega)$ and $V = W = H_0^1(\Omega)$ and define

$$\Phi(r) = \int_0^r M(s)ds.$$

Then putting

$$F_A(u) = \frac{1}{2}\Phi\left(\int_\Omega |\text{grad } u(x)|^2 dx\right),$$

we can apply Theorem 2.1. In this example, it is easily seen that the equality in (2.4) holds true. Furthermore, if M is a $C^1[0, \infty)$-function satisfying $M(0) > 0$, we can derive the uniqueness of weak solutions.

References

[1] V. Barbu: Nonlinear Semigroups and Differential Equations in Banach Spaces. Noordhoff International Publ. (1976).

[2] H. Brézis: Problèms unilatéraux. J. Math. Pures Appl., **51**, 1–168 (1972).

[3] R. W. Dickey: Infinite systems of nonlinear oscillation equations related to the string. Proc. Amer. Math. Soc., **23**, 459–468 (1969).

[4] R. J. Knops and B. Straughan: Non-existence of global solutions to nonlinear Cauchy problems arising in mechanics. Trends in Applications of Pure Mathematics to Mechanics (ed. by G. Fichera) Pitman Publ. (1976).

[5] L. A. Medeiros: Sur une équation non linéaire de la physique mathématique. C. R. Acad. Sc. Paris, **286**, 277–278 (1978).

[6] S. I. Pohozaev: On a class of quasilinear hyperbolic equations. Math. USSR Sbornik, **25**, 145–158 (1975).

[7] R. T. Rockafellar: Local boundedness of nonlinear, monotone operators. Michigan Math. J., **16**, 397–407 (1969).

[8] M. Tsutsumi: Some nonlinear evolution equations of second order. Proc. Japan Acad., **47**, 950–955 (1971).

[9] Y. Yamada: On the decay of solutions for some nonlinear evolution equations of second order. Nagoya Math. J., **73**, 69–98 (1979).