37. A Version of the Central Limit Theorem for Martingales

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§1. **Introduction.** In the present note let $\{X_n, \mathcal{F}_n\}$ be a zero-mean square-integrable martingale on a probability space $(\Omega, \mathcal{F}, P)$ and let $Y_1=X_1$, $Y_n=X_n-X_{n-1}$ $(n\geqq 2)$. Then our purpose is to prove the following

Theorem. *Suppose that there exist a sequence* $\{A_n\}$ *of positive numbers for which* $\lim_{n\to+\infty} A_n=+\infty$ *and a random variable* $Z(\omega)$ *such that*

(L-I) *for any given* $\varepsilon>0$, $\lim_{n\to+\infty} A_n^{-2}\sum_{k=1}^n E\{Y_k^2 I(|Y_k|\geqq \varepsilon A_n)\}=0$,*)

(L-II) $\lim_{n\to+\infty} A_n^{-2}\sum_{k=1}^n Y_k^2=Z$, *in probability.*

Then for any set $F\in\sigma(\bigcup_{n=1}^\infty \mathcal{F}_n)$ *and any real number* x $(x\neq 0)$

$$\lim_{n\to+\infty} P\{F, X_n(\omega)/A_n\leqq x\}=(2\pi)^{-1/2}\int_F\left\{\int_{-\infty}^{x/\sqrt{Z}}\exp(-u^2/2)du\right\}dP,$$

where $\sigma(\bigcup_{n=1}^\infty \mathcal{F}_n)$ *denotes the* σ*-algebra generated by the algebra* $\bigcup_{n=1}^\infty \mathcal{F}_n$ *and* $x/0$ *is* $+\infty$ (*or* $-\infty$) *if* x *is positive* (*or negative*).

In the important special case when Y_n's are independent and $\mathcal{F}_n$ is the σ-algebra generated by $\{X_k, k\leqq n\}$ the condition (L-I) for $A_n^2=EX_n^2$ is called Lindeberg's condition for the central limit theorem and in this case (L-I) implies (L-II) with $Z(\omega)=1$. But in general (L-I) does not imply (L-II) and even if the conditions (L-I) and (L-II) are satisfied the limit Z is not necessarily a constant. When $Z(\omega)$ is a constant, the central limit theorems are proved by many authors (cf. [1]).

As an application of Theorem we can prove the central limit theorem for $\{X_n\}$. In fact we prove the following

Corollary. *Under the conditions* (L-I) *and* (L-II) *if* $P\{Z(\omega)\neq 0\}>0$, *then we have for any real number* x

$$\lim_{n\to+\infty} P\{X_n(\omega)/A_n\leqq x\sqrt{Z(\omega)}\,|\,Z(\omega)\neq 0\}=(2\pi)^{-1/2}\int_{-\infty}^{x}\exp(-u^2/2)du.$$

In §2 we prove Theorem. By Lévy's continuity theorem it is enough to show that, for any fixed real number λ,

$$\lim_{n\to+\infty}\int_F \exp(i\lambda X_n/A_n)dP=\int_F \exp(-\lambda^2 Z/2)dP. \tag{1.1}$$

The right hand side of the above formula is the Fourier-Stieltjes transform of the function $(2\pi)^{-1/2}\int_F\left\{\int_{-\infty}^{x/\sqrt{Z}}\exp(-u^2/2)du\right\}dP$, $-\infty<x$

*) $I(A)$ denotes the indicator of the set A.

$<+\infty$.

§2. **Proof of Theorem.** By the condition (L-I) there exists a sequence $\{\varepsilon_n\}$ of positive numbers such that

$$(2.1)\qquad \begin{cases} 2\varepsilon_m \leqq 1 \text{ for all } m,\ \varepsilon_n \to 0,\ \sum_{k=1}^n P(|Y_k| \geqq \varepsilon_n A_n) \to 0 \quad \text{and} \\ A_n^{-2} \sum_{k=1}^n E\{Y_k^2 I(|Y_k| \geqq \varepsilon_n A_n)\} \to 0, \qquad \text{as } n \to +\infty. \end{cases}$$

Using this sequence $\{\varepsilon_n\}$ let us put for $k \geqq 1$ and $n \geqq 1$

$$Y_{k,n} = Y_k I(|Y_k| < \varepsilon_n A_n) - E\{Y_k I(|Y_k| < \varepsilon_n A_n) | \mathcal{F}_{k-1}\}.$$

Then for each n, $\{Y_{k,n}, \mathcal{F}_k, k \geqq 1\}$ is a martingale difference sequence.

Lemma 1. *We have*

(i) $\lim_{n\to+\infty} A_n^{-1} \sum_{k=1}^n |Y_k - Y_{k,n}| = 0$, *in pr.*,

(ii) $\lim_{n\to+\infty} A_n^{-2} \sum_{k=1}^n Y_{k,n}^2 = Z$, *in pr.*

Proof. Since $|E\{Y_k I(|Y_k| < \varepsilon_n A_n) | \mathcal{F}_{k-1}\}| = |E\{Y_k I(|Y_k| \geqq \varepsilon_n A_n) | \mathcal{F}_{k-1}\}|$, we have by (2.1)

$$\begin{aligned} A_n^{-1} \sum_{k=1}^n E|Y_k - Y_{k,n}| &\leqq 2A_n^{-1} \sum_{k=1}^n E|Y_k I(|Y_k| \geqq \varepsilon_n A_n)| \\ &\leqq 2A_n^{-1} \sum_{k=1}^n \{P(|Y_k| \geqq \varepsilon_n A_n) E Y_k^2 I(|Y_k| \geqq \varepsilon_n A_n)\}^{1/2} \\ &\leqq 2\left\{\sum_{k=1}^n P(|Y_k| \geqq \varepsilon_n A_n)\right\}^{1/2} \left\{A_n^{-2} \sum_{k=1}^n E Y_k^2 I(|Y_k| \geqq \varepsilon_n A_n)\right\}^{1/2} \to 0, \\ &\qquad\qquad \text{as } n \to +\infty, \end{aligned}$$

and we can prove the first part. Next we have

$$A_n^{-2} \sum_{k=1}^n |Y_k^2 - Y_{k,n}^2| \leqq \max_{1\leqq k\leqq n} (|Y_k| + |Y_{k,n}|) A_n^{-2} \sum_{k=1}^n |Y_k - Y_{k,n}|.$$

Therefore, we can prove (ii), by (2.1), (i) and (L-II).

Now for any fixed positive number M and $n \geqq 1$ let us put

$$S_n(\omega, M) = S_n(\omega) = \begin{cases} +\infty, & \text{if } \sum_{k=1}^\infty Y_{k,n}^2(\omega) \leqq M A_n^2, \\ \min\{m\,;\ \sum_{k=1}^m Y_{k,n}^2(\omega) > M A_n^2\}, & \text{otherwise.} \end{cases}$$

Then for each n, $S_n(\omega)$ is a stopping time with respect to $\{\mathcal{F}_k\}$ and

$$(2.2)\qquad S_n(\omega) \geqq M / 4\varepsilon_n^2$$

because $|Y_{k,n}(\omega)| \leqq 2\varepsilon_n A_n$. Next we put, for $k = 1, 2, \cdots, n$ and $n = 1, 2, \cdots$,

$$(2.3)\qquad \hat{Y}_{k,n} = Y_{k,n} I(S_n \geqq k) \quad \text{and} \quad \mathcal{F}_{k,n} = \mathcal{F}_{\min(k, S_n)}.$$

Then for each n, $\{\hat{Y}_{k,n}, \mathcal{F}_{k,n}, k \geqq 1\}$ is a martingale difference sequence (cf. [2, p. 300]) and by (2.1) and (2.3), we have

$$(2.4)\qquad A_n^{-2} \sum_{k=1}^n \hat{Y}_{k,n}^2 \leqq M + 4\varepsilon_n^2 \leqq M + 1.$$

In the following let λ denote any fixed real number and

$$P_{k,n}(\omega, \lambda) = P_{k,n}(\omega) = \prod_{j=1}^k (1 + i\lambda \hat{Y}_{j,n}(\omega) A_n^{-1}).$$

Lemma 2. *We have, for any set* $F \in \sigma(\bigcup_{n=1}^\infty \mathcal{F}_n)$,

$$\lim_{n\to+\infty} \int_F P_{n,n}(\omega) dP = P(F).$$

Proof. Since $|P_{k,n}|^2 \leqq \exp(\lambda^2 A_n^{-2} \sum_{j=1}^k \hat{Y}_{j,n}^2)$, (2.4) implies that

$$(2.5)\qquad |P_{k,n}|^2 \leqq \exp\{\lambda^2(M+1)\}, \qquad \text{for } 1 \leqq k \leqq n \text{ and } n \geqq 1.$$

On the other hand from the theory of measure it follows that if $F \in \sigma(\bigcup_{n=1}^{\infty} \mathcal{F}_n)$, then for any given $\varepsilon > 0$ there exists a set G such that $P\{F \Delta G\} < \varepsilon$ and $G \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$. Therefore, by (2.5) and the above fact it suffices to prove the lemma for any set $F \in \mathcal{F}_m$ where m is any fixed positive integer. Hereafter we assume that $F \in \mathcal{F}_m$. Then by (2.2) it is seen that

(2.6) $\quad F \in \mathcal{F}_{k,n}$, for all (k, n) such that $m \leqq k$ and $m \leqq M/4\varepsilon_n^2$.

Hence by (2.1), (2.6) and (2.5), we have for sufficiently large n

$$\begin{aligned}\int_F P_{n,n}(\omega)dP &= \int_F \left\{1 + \sum_{k=1}^{n} i\lambda \hat{Y}_{k,n}(\omega) A_n^{-1} P_{k-1,n}(\omega)\right\} dP \\ &= P(F) + \sum_{k=1}^{m} \int_F i\lambda \hat{Y}_{k,n}(\omega) A_n^{-1} P_{k-1,n}(\omega) dP \\ &= P(F) + O(m\,|\varepsilon_n \lambda|(1 + |2\varepsilon_n \lambda|)^{m-1}) \\ &= P(F) + o(1), \qquad \text{as } n \to +\infty.\end{aligned}$$

Lemma 3. *We have, for any set* $F \in \sigma(\bigcup_{n=1}^{\infty} \mathcal{F}_n)$,

$$\lim_{n \to +\infty} \int_F \exp\left(i\lambda \sum_{k=1}^{n} \hat{Y}_{k,n}/A_n\right) dP = \int_F \exp(-\lambda^2 Z_M/2) dP,$$

where $Z_M(\omega) = \min\{Z(\omega), M\}$.

Proof. From (2.3) and (2.1) it is seen that

$$A_n^{-2} \sum_{k=1}^{n} \hat{Y}_{k,n}^2 = A_n^{-2} \sum_{k=1}^{n} Y_{k,n}^2, \qquad \text{if } S_n(\omega, M) \geqq n,$$

$$M < A_n^{-2} \sum_{k=1}^{n} \hat{Y}_{k,n}^2 \leqq M + 4\varepsilon_n^2, \qquad \text{if } S_n(\omega, M) < n.$$

Therefore, (ii) in Lemma 1 and (2.1) imply that

$$\lim_{n \to +\infty} A_n^{-2} \sum_{k=1}^{n} \hat{Y}_{k,n}^2 = Z_M, \qquad \text{in pr.}$$

Hence, by (2.5)

$$\lim_{n \to +\infty} E\left|P_{n,n}\left\{\exp\left(-\lambda^2 \sum_{k=1}^{n} \hat{Y}_{k,n}^2/2A_n^2\right) - \exp(-\lambda^2 Z_M/2)\right\}\right| = 0.$$

On the other hand since Lemma 2 and (2.5) imply that

$$\lim_{n \to +\infty} \int_F P_{n,n} \exp(-\lambda^2 Z_M/2) dP = \int_F \exp(-\lambda^2 Z_M/2) dP,$$

we have

(2.7) $$\lim_{n \to +\infty} \int_F P_{n,n} \exp\left(-\lambda^2 \sum_{k=1}^{n} \hat{Y}_{k,n}^2/2A_n^2\right) dP = \int_F \exp(-\lambda^2 Z_M/2) dP.$$

Further it is easily seen that if $|x| < 1/2$, then

$$\exp(x) = (1 + x)\exp\{(x^2/2) + \theta(x)\} \quad \text{and} \quad |\theta(x)| \leqq |x|^3.$$

Therefore, by (2.1) and (2.4)

$$\left|\sum_{k=1}^{n} \theta(i\lambda \hat{Y}_{k,n}/A_n)\right| \leqq 2\varepsilon_n |\lambda|^3 (M + 1) = o(1), \qquad \text{as } n \to +\infty,$$

and since $|P_{n,n} \exp(-\lambda^2 \sum_{k=1}^{n} \hat{Y}_{k,n}^2/2A_n^2)| \leqq 1$, we have

$$\begin{aligned}\exp\left(i\lambda \sum_{k=1}^{n} \hat{Y}_{k,n} A_n^{-1}\right) &= P_{n,n} \exp\left\{-\lambda^2 \sum_{k=1}^{n} \hat{Y}_{k,n}^2/2A_n^2 + \sum_{k=1}^{n} \theta(i\lambda \hat{Y}_{k,n}/A_n)\right\} \\ &= P_{n,n} \exp\left(-\lambda^2 \sum_{k=1}^{n} \hat{Y}_{k,n}^2/2A_n^2\right) + o(1),\end{aligned}$$

uniformly on Ω, as $n \to +\infty$.

Thus by the above relation and (2.7) we can prove the lemma.

Lemma 4. *We have, for any set* $F \in \sigma(\bigcup_{n=1}^{\infty} \mathcal{F}_n)$,

$$\lim_{n\to+\infty} \int_F \exp\left(i\lambda \sum_{k=1}^{n} Y_k/A_n\right)dP = \int_F \exp(-\lambda^2 Z/2)dP.$$

Proof. By (i) in Lemma 1 it is enough to show that

$$\lim_{n\to+\infty} \int_F \exp\left(i\lambda \sum_{k=1}^{n} Y_{k,n}/A_n\right)dP = \int_F \exp(-\lambda^2 Z/2)dP. \tag{2.8}$$

If we put $E_M=\{Z(\omega)>M\}$ and $E_{n,M}=\{A_n^{-2}\sum_{k=1}^{n} Y_{k,n}^2(\omega)>M\}$, then (ii) in Lemma 1 implies that $P(E_{n,M})\to P(E_M)$ as $n\to+\infty$, at the continuity points M of $P(E_M)$. Therefore, for any given $\varepsilon>0$ we can take M and n_0 such that

$$P(E_M)<\varepsilon \quad \text{and} \quad P(E_{n,M})<\varepsilon \qquad \text{if } n\geqq n_0.$$

Since $\omega \in E_{n,M}^c$ and $k\leqq n$ imply that $\hat{Y}_{k,n}(\omega)=Y_{k,n}(\omega)$, we have for $n\geqq n_0$

$$\begin{cases} E\left|\exp(i\lambda \sum_{k=1}^{n} \hat{Y}_{k,n}/A_n)-\exp(i\lambda \sum_{k=1}^{n} Y_{k,n}/A_n)\right|<2\varepsilon, \\ E\left|\exp(-\lambda^2 Z_M/2)-\exp(-\lambda^2 Z/2)\right|<\varepsilon. \end{cases}$$

Thus by Lemma 3 and above relations we can prove (2.8).

By Lemma 4 (1.1) holds and Theorem is proved.

§ 3. Proof of Corollary. For simplicity of writing the formulas we prove Corollary only for positive x. Let ε $(0<\varepsilon<1)$ be any given number and put for $k=0, \pm 1, \pm 2, \cdots$ and h $(0<2h<\varepsilon)$

$$a(k)=\exp(kh) \quad \text{and} \quad E_k=\{a(k)\leqq \sqrt{Z(\omega)}<a(k+1)\}.$$

Then clearly E_k's are disjoint sets in $\sigma(\bigcup_{n=1}^{\infty}\mathcal{F}_n)$ and $\bigcup_{k=-\infty}^{\infty} E_k=\{Z(\omega)\neq 0\}$. Therefore we have, by Theorem,

$$\begin{aligned} &\overline{\lim_{n\to+\infty}}\, P\{X_n/A_n\leqq x\sqrt{Z}, Z\neq 0\} \\ &\quad\leqq \sum_{|k|<m_0} \lim_{n\to+\infty} P\{X_n/A_n\leqq xa(k+1), E_k\}+\sum_{|k|\geqq m_0} P(E_k) \\ &\quad\leqq \sum_{k=-\infty}^{\infty} (2\pi)^{-1/2}\int_{E_k}\left\{\int_{-\infty}^{xa(k+1)/\sqrt{Z}} \exp(-u^2/2)du\right\}dP+\sum_{|k|\geqq m_0} P(E_k), \end{aligned}$$

and in the same way

$$\begin{aligned} &\underline{\lim_{n\to+\infty}}\, P\{X_n/A_n\leqq x\sqrt{Z}, Z\neq 0\} \\ &\quad\geqq \sum_{k=-\infty}^{\infty} (2\pi)^{-1/2}\int_{E_k}\left\{\int_{-\infty}^{xa(k)/\sqrt{Z}} \exp(-u^2/2)du\right\}dP-\sum_{|k|\geqq m_0} P(E_k). \end{aligned}$$

Since $x\exp(-x^2/2)<1$ for $x>0$, we have

$$\begin{aligned} &\sum_k \int_{E_k}\left\{\int_{xa(k)/\sqrt{Z}}^{xa(k+1)/\sqrt{Z}} \exp(-u^2/2)du\right\}dP \\ &\quad\leqq \sum_k \int_{E_k} (e^h-1)(xa(k)/\sqrt{Z})\exp\{-x^2a^2(k)/2Z\}dP \\ &\quad\leqq \sum_k 2hP(E_k)<\varepsilon P(Z\neq 0)<\varepsilon. \end{aligned}$$

Since $xa(k)/\sqrt{Z}\leqq x<xa(k+1)/\sqrt{Z}$ on E_k and we can take m_0 so large that $\sum_{|k|\geqq m_0} P(E_k)<\varepsilon$, Corollary is proved.

References

[1] B. M. Brown: Martingale central limit theorems. Ann. Math. Stats., **42**, 59–66 (1971).

[2] J. L. Doob: Stochastic Processes. John Wiley and Sons, New York (1953).