26. *The Hodge Conjecture and the Tate Conjecture for Fermat Varieties*

By Tetsuji SHIODA
Department of Mathematics, University of Tokyo

(Communicated by Kunihiko KODAIRA, M. J. A., March 12, 1979)

Throughout the paper, $X_m^n(p)$ will denote the Fermat variety of dimension n and of degree m in characteristic p ($p=0$ or a prime number not dividing m), defined by the equation

$$x_0^m + x_1^m + \cdots + x_{n+1}^m = 0. \tag{1}$$

The purpose of this note is to report our results on the Hodge Conjecture for $X_m^n(0)$ and the Tate Conjecture for $X_m^n(p)$, $p>0$. By means of the inductive structure of $X_m^n(p)$ with respect to n ([3, § 1]), we can reduce the proof of these conjectures to the verification of certain purely arithmetic conditions on m, n and p. After formulating the condition in § 1, we state the main results in §§ 2 and 3. We give the brief sketch of the proof in § 4.

Detailed accounts will be published elsewhere.

§ 1. The arithmetic condition. Fix $m>1$, and let H be a cyclic subgroup of order f of $(\boldsymbol{Z}/m)^\times$. We consider the following system of linear Diophantine equations in $x_1, \cdots, x_{m-1}$ and y

$$\sum_{\nu=1}^{m-1} \sum_{u \in H} \langle tu\nu \rangle x_\nu = fmy \qquad \text{for all } t \in (\boldsymbol{Z}/m)^\times, \tag{2}$$

where, for $a \in \boldsymbol{Z}/m - \{0\}$, $\langle a \rangle$ denotes the representative of a between 1 and $m-1$. Let $M_m(H)$ denote the additive semigroup of non-negative integer solutions $(x_1, \cdots, x_{m-1}; y)$ of (2) satisfying moreover the following congruence:

$$\sum_{\nu=1}^{m-1} \nu x_\nu \equiv 0 \qquad (\text{mod } m). \tag{3}$$

For an element $\xi = (x_1, \cdots, x_{m-1}; y)$ of $M_m(H)$, we call y the length of ξ and write $y = \|\xi\|$. (We exclude the trivial solution $(0, \cdots, 0; 0)$.) If H' is a cyclic subgroup of H, then $M_m(H')$ is contained in $M_m(H)$; in particular, setting $M_m = M_m(\{1\})$, we have $M_m \subset M_m(H)$ for any H. There are exactly $[m/2]$ elements of length 1 in $M_m(H)$ and they are all contained in M_m.

Definition. Let $\xi = (x_1, \cdots, x_{m-1}; y) \in M_m(H)$. Then

(i) ξ is *decomposable* if $\xi = \xi' + \xi''$ for some $\xi', \xi'' \in M_m(H)$; otherwise ξ is called *indecomposable*.

(ii) ξ is *quasi-decomposable* if there exists $\eta \in M_m(H)$ with $\|\eta\| \leq 2$ such that $\xi + \eta = \xi' + \xi''$ for some $\xi', \xi'' \in M_m(H)$ with $\|\xi'\|, \|\xi''\| < \|\xi\|$.

(iii) ξ is *semi-decomposable* if there exist non-negative integer solutions (x'_ν) and (x''_ν) of (3) such that $x_\nu = x'_\nu + x''_\nu$ and $\sum x'_\nu = \sum x''_\nu = 3$ (this occurs only if $y = \|\xi\| = 3$).

By Gordan's lemma, there are only finitely many indecomposable elements in $M_m(H)$, and they form the minimal set of generators of $M_m(H)$. Now let us formulate the following conditions $(P^n_m(H))$ for n even:

$(P^n_m(H))$ Every indecomposable element ξ of $M_m(H)$ with $3 \le \|\xi\| \le n/2+1$ is either quasi-decomposable or semi-decomposable.

This condition is vacuous if $n \le 2$ or if $M_m(H)$ has no indecomposable elements with length ≥ 3. For sufficiently large n, $(P^n_m(H))$ is equivalent to the following:

$(P_m(H))$ $M_m(H)$ has no indecomposable elements of length ≥ 3 which are neither quasi-decomposable nor semi-decomposable.

§2. The Hodge Conjecture for $X^n_m(0)$. Given a smooth projective variety X over the field of complex numbers $\boldsymbol{C}$, the Hodge Conjecture for X states that the space of rational cohomology classes of type (d, d) on X is spanned over $\boldsymbol{Q}$ by the classes of algebraic cycles of codimension d on X (cf. [1]). For the Fermat variety $X^n_m = X^n_m(0)$ over $\boldsymbol{C}$, this is non-trivial only in case n is even and $d = n/2$. We call the condition $(P^n_m(H))$ or $(P_m(H))$ for $H = \{1\}$ simply (P^n_m) or (P_m).

Theorem 1. *If the condition (P^n_m) is satisfied, then the Hodge Conjecture for the Fermat variety X^n_m is true.*

The condition (P^n_m) has been verified for the following values of m and n (at least): 1) m prime, all n (Parry), 2) $m \le 20$, all n and 3) $m = 21$ and $n \le 10$. Therefore the Hodge Conjecture for X^n_m is true for these m and n. Thus we have extended the recent results of Ran [2] for m prime to some extent. Hopefully the condition (P^n_m) might be always true.

Theorem 2. *Fix $m > 1$. If the condition (P_m) is satisfied, then the Hodge Conjecture for arbitrary product $X^{n_1}_m \times \cdots \times X^{n_k}_m$ is true.*

§3. The Tate Conjecture for $X^n_m(p)$. Given a smooth projective variety X over a finite field $k = \boldsymbol{F}_q$ such that $\bar{X} = X \times_k \bar{k}$ is irreducible ($\bar{k}$ = the algebraic closure of k), the Tate Conjecture for X over k states that the order of pole of the zeta function $Z(X/k, T)$ at $T = 1/q^d$ is equal to the dimension of the subspace of $H^{2d}_{et}(\bar{X}, \boldsymbol{Q}_l)$ spanned by classes of k-rational algebraic cycles of codimension d on X ([5, §3]). For the Fermat variety $X^n_m(p)$, this is non-trivial only in case n is even and $d = n/2$.

We choose the base field $k = \boldsymbol{F}_q$ for $X^n_m(p)$ as follows. Let f be the order of p mod m in $(\boldsymbol{Z}/m)^\times$ and let $q = p^{fm'}$, where $m' =$ L.C.M. $(m, 2)$. We denote by H_p the cyclic subgroup of $(\boldsymbol{Z}/m)^\times$ generated by p mod m, and call the condition $(P^n_m(H_p))$ or $(P_m(H_p))$ simply $(P^n_m(p))$ or $P_m(p))$.

Theorem 3. *With the above notation, the Tate Conjecture for $X_m^n(p)$ over F_q is true, provided that the condition $(P_m^n(p))$ is satisfied.*

The condition $(P_m^n(p))$ has been verified in the following cases:

i) $p\equiv 1 \pmod m$, m, n satisfying (P_m^n) (cf. § 2).

ii) $p^\nu \equiv -1 \pmod m$ for some ν, m, n arbitrary ("supersingular" case).

Tate himself proved the Conjecture in case i) with $n=2$ and in case ii), and remarked that the case i) with arbitrary n (even) follows from the Hodge Conjecture for X_m^n ([5, p. 102]). We have also proved the Tate Conjecture for $X_m^n(p)$ in case ii) and in the surface case:

iii) $n=2$, p, m arbitrary ([3, § 2]).

Furthermore, we have verified the condition $(P_m^n(p))$ in a few more cases:

iv) $m\leq 8$, p, n arbitrary.

Note that some cases in iv) are not covered by i), ii) or iii), i.e. $n>2$ and $m=7$, $p\equiv 2, 4$ (7) or $m=8$, $p\equiv 3, 5$ (8).

Theorem 4. *Fix m and p. If the condition $(P_m(p))$ is satisfied, then the Tate Conjecture for arbitrary product $X_m^{n_1}\times\cdots\times X_m^{n_k}$ is true.*

Remark. The global Tate Conjecture for X_m^n over certain algebraic number fields follows from the Hodge Conjecture for X_m^n (cf. [5, § 4]).

§ 4. The outline of the proof. We shall briefly outline the basic idea of the proof. For simplicity, we write $X^n = X_m^n(p)$, fixing m and p. Let $n=r+s$ with $r, s\geq 1$. Using the inductive structure of X^n ([3, Theorem 1.7]), we have a natural isomorphism

$$(*)\quad [H^r_{\mathrm{prim}}(X^r)\otimes H^s_{\mathrm{prim}}(X^s)]^{\mu_m}\oplus[H^{r-1}_{\mathrm{prim}}(X^{r-1})\otimes H^{s-1}_{\mathrm{prim}}(X^{s-1})(1)]\xrightarrow{\sim} H^n_{\mathrm{prim}}(X^n),$$

which is equivariant with respect to the natural action of G^n on each term and which preserves algebraic cycles. Here G^n is the quotient group of the $(n+2)$-fold product of μ_m by the subgroup of diagonal elements, and $H^n_{\mathrm{prim}}(X^n)$ is the "primitive part" of $H^n(X^n)$ if n is even $(n\geq 0)$, and equal to $H^n(X^n)$ if n is odd. The cohomology $H^n(X^n)$ is the complex cohomology if $p=0$, and the l-adic etale cohomology if $p>0$, where l is a prime number such that $l\neq p$ and $l\equiv 1 \pmod m$. We have the eigenspace decomposition of $H^n_{\mathrm{prim}}(X^n)$:

$$H^n_{\mathrm{prim}}(X^n)=\bigoplus_{\alpha\in\mathfrak{A}_m^n} V(\alpha),\qquad \dim V(\alpha)=1,$$

where $\mathfrak{A}_m^n$ is the subset of characters of G^n defined by

$$\mathfrak{A}_m^n=\{\alpha=(a_0,\cdots,a_{n+1})\,|\,a_i\in Z/m,\ a_i\neq 0,\ \textstyle\sum a_i=0\}.$$

If $p>0$, the decomposition is compatible with the action of Frobenius endomorphism F of X^n relative to F_q; the eigenvalue of F^* on $V(\alpha)$ is given by the Jacobi sum $j(\alpha)$ of Weil [7] up to the sign $(-1)^n$. The condition for $j(\alpha)$ to contribute to the pole of $Z(X^n/F_q, T)$ can be explicitly described by Stickelberger's theorem ([8], cf. [3]). If $p=0$, the condition for $V(\alpha)$ to come from rational cohomology classes of type $(n/2, n/2)$ can also be described by α ([2], [4]).

Now, by the map (*), we can construct algebraic cycles on X^n from those on $X^r \times X^s$ or $X^{r-1} \times X^{s-1}$. The conditions (P_m^n) or $(P_m^n(p))$ say exactly when every candidate of algebraic cycles on $X_m^n(0)$ or $X_m^n(p)$ can be constructed inductively from algebraic cycles on X^0, X^2 or $X^1 \times X^1$, where the Hodge Conjecture or the Tate Conjecture is known, the former by Lefschetz theorem and the latter by Tate [6] and Shioda-Katsura [3]. This proves Theorems 1 and 3.

The proof of Theorems 2 and 4 also depends on the existence of the isomorphism (*) preserving algebraic cycles.

References

[1] W. V. D. Hodge: The topological invariants of algebraic varieties. Proc. Int. Congr. Math., 181–192 (1950).

[2] Z. Ran: Cycles on Fermat hypersurfaces (preprint).

[3] T. Shioda and T. Katsura: On Fermat varieties. Tôhoku Math. J., **31**, 97–115 (1979).

[4] T. Shioda: The Hodge Conjecture for Fermat varieties.

[5] J. Tate: Algebraic cycles and poles of zeta functions. Arithmetical Algebraic Geometry, Harper and Row, New York, 93–110 (1965).

[6] ——: Endomorphisms of abelian varieties over finite fields. Invent. Math., **2**, 134–144 (1966).

[7] A. Weil: Numbers of solutions of equations in finite fields. Bull. Amer. Math. Soc., **55**, 497–508 (1949).

[8] ——: Jacobi sums as "Grössencharaktere". Trans. Amer. Math. Soc., **73**, 487–495 (1952).