24. *The Spectrum of the Laplacian and Boundary Perturbation. I*

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Introduction. Let Ω be a bounded domain in $\boldsymbol{R}^2$ with smooth boundary γ. We assume, for simplicity, that Ω is simply connected. Consider eigenvalue problem for the Laplacian under Dirichlet condition

$$(1)\qquad \begin{cases} (-\Delta-\lambda)u(x)=0 & x\in\Omega \\ \qquad\qquad u|_\gamma=0. \end{cases}$$

Let $0<\lambda_1\leq\lambda_2\leq\lambda_3\leq\cdots$ be the eigenvalues of the problem (1). These are functions of γ. The totality of the boundaries γ, with appropriate regularity, forms a separable Hilbert manifold Γ. A subset of Γ is called residual if it is a countable intersection of open dense subsets of Γ. Our main theorem is

Theorem 1. *There is a residual subset B of Γ such that for any $\gamma\in B$ all the eigenspaces of the problem* (1) *are of dimension one.*

Since the complement of B is a set of first category, Theorem 1 means that for generic γ the eigenvalues of the Laplacian are all simple. Similar results were already obtained by Uhlenbeck [4] in the case of potential perturbation or in the case that Δ is the Laplace Beltrami operator on a compact Riemannian manifold and the perturbation is that of the metric.

Theorem 1 was conjectured by Arnold [1]. But the proof was not given as far as the present authors know. Our proof can easily be generalized to the case that Ω is a domain of $\boldsymbol{R}^n$.

§ 1. The Hilbert manifold Γ of boundary curves. Let S^1 be the unit circle$=\{e^{i\theta}\mid 0\leq\theta\leq 2\pi\}$. Let Γ^k $(k\geq 1)$ be the totality of embeddings

$$\gamma;\quad S^1\ni e^{i\theta}\longrightarrow\gamma(\theta)=(x_1(\theta),x_2(\theta))\in\boldsymbol{R}^2$$

such that the functions $x_1(\theta)$ and $x_2(\theta)$ belong to the Sobolev space $H^k(S^1)$ of order k. $\gamma(\theta)$ is of class $C^{k-1}(S^1)$ by virtue of Sobolev embedding theorem. Let $\gamma'\in\Gamma^k$ and let $\gamma'(\theta)=(x_1'(\theta),x_2'(\theta))$. Then we put

$$(1.1)\qquad \rho(\gamma,\gamma')=(\|x_1-x_1'\|_k^2+\|x_2-x_2'\|_k^2)^{1/2},$$

where $\|\ \|_k$ denotes the Sobolev norm of order k. We can easily see that Γ^k is a separable Hilbert manifold and that $\rho(\gamma,\gamma')$ is a metric compatible with this structure.

In the following, we fix $k\geq 5$ and abbreviate Γ^k as Γ. Let $\gamma\in\Gamma$. Then γ is a simple Jordan curve of class C^{k-1} and γ bounds a bounded simply connected domain Ω_γ in $\boldsymbol{R}^2$. The unit outer normal vector $\nu(\theta)$ to γ at $\gamma(\theta)$ is of class C^2. Tubular neighbourhood theorem holds for γ. Hence, there exists a positive constant $\varepsilon(\gamma)$ such that, for any $\gamma' \in \Gamma$ satisfying $\rho(\gamma, \gamma') < \varepsilon(\gamma)$, there exists a diffeomorphism $\omega_\gamma^{\gamma'}$; $\bar{\Omega}_\gamma \to \bar{\Omega}_{\gamma'}$ of class C^4 whose restriction to γ coincides with $\gamma' \circ \gamma^{-1}$. Moreover, taking $\varepsilon(\gamma)$ smaller if necessary, we may assume that the correspondence $(\gamma, \gamma') \to \omega_\gamma^{\gamma'}$ is of class C^4. Therefore, if γ_t, $t \in [0,1]$, is a C^1 curve in Γ starting at $\gamma \equiv \gamma_0$, then we have one parameter family of mappings $\{\omega_\gamma^{\gamma_t}\}_{t\in[0,1]}$. At every $x \in \bar{\Omega}_\gamma$, we put

$$X(x) = \frac{\partial \omega_\gamma^{\gamma_t}(x)}{\partial t}\bigg|_{t=0}. \tag{1.2}$$

Then $X(x) = (X_1(x), X_2(x))$ is a vector field defined in $\bar{\Omega}_\gamma$. Since $\omega_\gamma^{\gamma'}$ is not uniquely determined, the vector field $X(x)$ is not uniquely determined by the curve γ_t. However its restriction to γ, that is, $\delta\gamma(\theta) = X(\gamma(\theta))$ is uniquely determined by the curve γ_t in Γ. Thus the vector field $\theta \to \delta\gamma(\theta) = X(\gamma(\theta))$ is identified with the tangent vector to γ_t at $\gamma_0 = \gamma$. Thus we have

$$T_\gamma \Gamma = \{\delta\gamma(\theta) \mid \delta\gamma(\theta) = X(\gamma(\theta)) \in H^k(S^1) \times H^k(S^1)\}. \tag{1.3}$$

The normal components of $\delta\gamma(\theta)$ is given by

$$\{\delta\gamma(\theta), \nu_\theta\} = \langle X(\gamma(\theta)), \nu_\theta \rangle \tag{1.4}$$

where $\langle\ ,\ \rangle$ denotes Euclidean inner product in $\boldsymbol{R}^2$.

§2. **Main theorem.** Let $\gamma \in \Gamma$ and Ω_γ be as above. We consider problem (1) in Ω_γ. Since the manifold Γ is separable, Theorem 1 can be localized.

Theorem 1′. *At every $\tilde{\gamma} \in \Gamma$, there exist an open neighbourhood $U(\tilde{\gamma})$ of $\tilde{\gamma}$ and its residual subset $\boldsymbol{B}(\tilde{\gamma})$ such that for any $\gamma \in B(\tilde{\gamma})$ all the eigenspaces of problem* (1) *with $\Omega = \Omega_\gamma$ are of dimension one.*

Now we make a sketch of the proof of Theorem 1′. Let $g_\gamma(x, x')$ be the Green function for the Dirichlet problem in Ω_γ. The Green operator is defined by

$$G_\gamma u(x) = \int_{\Omega_\gamma} g_\gamma(x, x') u(x') dx'. \tag{2.1}$$

The eigenvalue problem (1) for $\Omega = \Omega_\gamma$ is transformed into

$$(I - \lambda G_\gamma) u(x) = 0. \tag{2.2}$$

We may consider this in $L^2(\Omega_\gamma)$. Let $U(\tilde{\gamma}) = \{\gamma \in \Gamma \mid \rho(\tilde{\gamma}, \gamma) < \varepsilon(\tilde{\gamma})\}$ and let $\gamma \in U(\tilde{\gamma})$. Then there is a C^4 diffeomorphism

$$\omega_{\tilde{\gamma}}^{\gamma};\ \bar{\Omega}_{\tilde{\gamma}} \longrightarrow \bar{\Omega}_\gamma$$

as stated in §1. For any $u \in L^2(\Omega_\gamma)$ the function $\omega_{\tilde{\gamma}}^{\gamma*} u(x) = u(\omega_{\tilde{\gamma}}^{\gamma}(x)) \in L^2(\Omega_{\tilde{\gamma}})$. Putting $\tilde{u}(x) = \omega_{\tilde{\gamma}}^{\gamma*} u(x)$ and $\tilde{v}(x) = \omega_{\tilde{\gamma}}^{\gamma*} v(x)$, we have

$$\int_{\Omega_\gamma} u(y) v(y) J_\gamma^{\tilde{\gamma}}(y) dy = \int_{\Omega_{\tilde{\gamma}}} \tilde{u}(x) \tilde{v}(x) dx, \tag{2.3}$$

where $J_\gamma^{\tilde{\gamma}}(y) =$ the Jacobian of the map $\omega_{\tilde{\gamma}}^{\gamma-1}$. The eigenvalue problem (2.2) is equivalent to

$$(I - \lambda G_{\tilde{\gamma}}^{\gamma}) \tilde{u}(x) = 0, \tag{2.4}$$

where $\tilde{u}(x)=(\omega_{\tilde{\gamma}}^{\gamma*}u)(x)$ and $G_{\tilde{\gamma}}^{\gamma}=\omega_{\tilde{\gamma}}^{\gamma*}G_{\gamma}(\omega_{\tilde{\gamma}}^{\gamma*})^{-1}$. Thus we consider problem (2.4) in the fixed Hilbert space $\mathfrak{h}=L^2(\Omega_{\tilde{\gamma}})$. Let

$$\mathfrak{S}=\left\{\tilde{u}\in L^2(\Omega_{\tilde{\gamma}})\,\middle|\,\int_{\Omega_{\tilde{\gamma}}}|\tilde{u}(x)|^2\,dx=1\right\}$$

be the unit sphere of $\mathfrak{h}$. Clearly, $\mathfrak{S}$ is a separable Hilbert manifold. We define the following map:

$$\begin{array}{lcc} \Phi\,;\ U(\tilde{\gamma})\times\mathfrak{S}\times\boldsymbol{R} & \longrightarrow & \mathfrak{h} \\ \quad\quad\ \ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ (\gamma,\tilde{u},\lambda) & \longrightarrow & \Phi(\gamma,\tilde{u},\lambda)=(\tilde{u}-\lambda G_{\tilde{\gamma}}^{\gamma}\tilde{u}). \end{array} \tag{2.5}$$

We can easily prove

Proposition 1. *The mapping Φ is a C^4 Fredholm mapping of index* 0.

Let $(\gamma,\tilde{u},\lambda)\in U(\tilde{\gamma})\times\mathfrak{S}\times\boldsymbol{R}$. Then, the differential of Φ at this point is

$$\delta\Phi(\gamma,\tilde{u},\lambda)=\delta\lambda G_{\tilde{\gamma}}^{\gamma}\tilde{u}+(I-\lambda G_{\tilde{\gamma}}^{\gamma})\delta\tilde{u}+\lambda(\delta G_{\tilde{\gamma}}^{\gamma})u. \tag{2.6}$$

The condition $\delta\tilde{u}\in T_{\tilde{u}}\mathfrak{S}$ is

$$\int_{\Omega_{\tilde{\gamma}}}\tilde{u}(x)\delta\tilde{u}(x)dx=0. \tag{2.7}$$

The last term of the right hand side of (2.6) is the variation caused by the boundary perturbation $\delta\gamma(\theta)=X(\gamma(\theta))\in T_{\gamma}\Gamma$. This term can be calculated explicitly.

Proposition 2 (Hadamard's variational formula). *For any $u\in L^2(\Omega_{\gamma})$, we have*

$$(\omega_{\tilde{\gamma}}^{\gamma*})^{-1}(\delta G_{\tilde{\gamma}}^{\gamma})\omega_{\tilde{\gamma}}^{\gamma*}u(y)=V_{\delta\gamma}u(y) \tag{2.8}$$

where

$$\begin{aligned} V_{\delta\gamma}u(y)=&-\int_{\Omega_{\gamma}}\int_{\gamma}\frac{\partial g_{\gamma}(y,\gamma(\theta))}{\partial\nu_{\theta}}\frac{\partial g_{\gamma}(\gamma(\theta),y')}{\partial\nu_{\theta}}\langle X(\gamma(\theta)),\nu_{\theta}\rangle d\sigma_{\theta}u(y')dy' \\ &+\langle\operatorname{grad}G_{\gamma}u(y),X(y)\rangle-G_{\gamma}(\langle X,\operatorname{grad}u\rangle)(y). \end{aligned} \tag{2.9}$$

Here $d\sigma_{\theta}$ is the line element of γ and $\langle X(\gamma(\theta)),\nu_{\theta}\rangle\in T_{\gamma}\Gamma$ as described in § 1. (For the proof of this, see [2], [3].)

Now we can prove,

Theorem 2. $\mathfrak{h}\ni 0$ *is the regular value of the mapping*

$$\Phi\,;\ U(\tilde{\gamma})\times\mathfrak{S}\times\boldsymbol{R}\longrightarrow\mathfrak{h}.$$

Proof. Assume that $\Phi(\gamma,\tilde{u},\lambda)=0\in\mathfrak{h}$. Then,

$$(I-\lambda G_{\tilde{\gamma}}^{\gamma})\tilde{u}=0. \tag{2.10}$$

Setting $u(x)=(\omega_{\tilde{\gamma}}^{\gamma*})^{-1}\tilde{u}$, we have

$$(I-\lambda G_{\gamma})u=0 \qquad \text{in } L^2(\Omega_{\gamma}). \tag{2.11}$$

We want to prove that the image of $\delta\Phi(\gamma,\tilde{u},\lambda)$ coincides with $\mathfrak{h}=L^2(\Omega)_{\tilde{\gamma}}$. Assume that $\tilde{v}\in\mathfrak{h}$ is orthogonal to the image of $\delta\Phi(\gamma,\tilde{u},\lambda)$. Then

$$\begin{aligned} 0=&\,\delta\lambda\int_{\Omega_{\tilde{\gamma}}}G_{\tilde{\gamma}}^{\gamma}\tilde{u}(x)\tilde{v}(x)dx+\int_{\Omega_{\tilde{\gamma}}}(I-\lambda G_{\tilde{\gamma}}^{\gamma})\delta\tilde{u}(x)\tilde{v}(x)dx \\ &+\lambda\int_{\Omega_{\tilde{\gamma}}}(\delta G_{\tilde{\gamma}}^{\gamma})\tilde{u}(x)\tilde{v}(x)dx. \end{aligned} \tag{2.12}$$

$\delta\tilde{u}\in J_{\tilde{u}}\mathfrak{S}$ satisfies (2.7) or equivalently, (2.13) $$\int_{\Omega_\gamma} u(y)\delta u(y) J_\gamma^\gamma(y)dy=0,$$

where $\delta\tilde{u}=(\omega_\gamma^{\gamma*})^{-1}\delta u$. The equation (2.12) is equivalent to the system of equations

(2.14) $$\int_{\Omega_\gamma} u(y)w(y)dy=0,$$

(2.15) $$\int_{\Omega_\gamma} ((I-\lambda G_\gamma)\delta\tilde{u})(y)w(y)dy=0,$$

(2.16) $$\int_{\Omega_\gamma} V_{\delta_\gamma}u(x)w(x)dx=0,$$

where we put

(2.17) $$w(y)=(\omega_\gamma^{\gamma*})^{-1}\tilde{v}(y)J_\gamma^\gamma(y).$$

Since $\delta\tilde{u}$ is arbitrary except for the condition (2.13), we obtain from (2.15) that

(2.18) $$(I-\lambda G_\gamma)w(y)=C\cdot u(y)J_\gamma^\gamma(y)$$

with some constant C. On the other hand $I-\lambda G_\gamma$ is symmetric in $L^2(\Omega_\gamma)$ and $u\in\ker(I-\lambda G_\gamma)$. Therefore,

$$0=\int_{\Omega_\gamma} u(y)(I-\lambda G_\gamma)w(y)dy=C\int_{\Omega_\gamma} |u(y)|^2 J_\gamma^\gamma(y)dy.$$

This yields that $C=0$ and

(2.19) $$(I-\lambda G_\gamma)w(y)=0 \qquad \text{in } L^2(\Omega_\gamma).$$

Consequently, $w(y)\in C^4(\bar{\Omega}_\gamma)$ and

(2.20) $$(-\Delta-\lambda)w(y)=0 \qquad \text{in } \Omega_\gamma$$
$$w|_\gamma=0.$$

As a consequence of (2.11), (2.19) and (2.16), we have

(2.21) $$\lambda^{-1}\int_\gamma \frac{\partial u}{\partial\nu_\theta}(\gamma(\theta))\frac{\partial w}{\partial\nu_\theta}(\gamma(\theta))\langle X(\gamma(\theta)),\nu_\theta\rangle d\sigma_\theta=0.$$

Since $\langle X(\gamma(\theta)),\nu_\theta\rangle=\delta\gamma(\theta)$ is arbitrary and $u\not\equiv 0$,

(2.22) $$\frac{\partial w}{\partial\nu_\theta}(\gamma(\theta))\equiv 0$$

in some open set of γ. It follows from this, (2.20) and Aronszajn's theorem, that

$$w(y)\equiv 0 \qquad \text{for } \forall y\in\Omega_\gamma.$$

Thus $\tilde{v}(x)\equiv 0$. This proves Theorem 2.

Theorem 1′ follows from Theorem 2 and a result of Uhlenbeck [4].

§ 3. Manifolds of operators with multiple eigenvalues. Let $\mathcal{L}_+^2$ = the totality of symmetric positive Hilbert-Schmidt operators. Then $\mathcal{L}_+^2$ is also a separable Hilbert manifold. For any $K\in\mathcal{L}_+^2$, let $\mu_1(K)\geq\mu_2(K)\geq\mu_3(K)\geq\cdots>0$ denote its eigenvalues. We put for any pair of positive integers l, m,

$$Q_{lm}=\{K\in\mathcal{L}_+^2 \mid \text{the eigenvalues of } K \text{ satisfy}$$
$$\mu_{l-1}(K)>\mu_l(K)=\mu_{l+1}(K)=\cdots=\mu_{l+m-1}(K)>\mu_{l+m}(K)\}.$$

Theorem 3. *For any pair of positive integers $l\geq 1$ and $m\geq 2$, Q_{lm} is a C^∞ submanifold of $\mathcal{L}_+^2$ of codimension* $(1/2)(m-1)(m+2)$.

Proof. Let K^0 be an arbitrary point in Q_{lm}. Let $\varphi_1, \varphi_2, \varphi_3, \cdots$ be eigenvectors of K^0. Let

E_0 = the vector space generated by $\varphi_l, \varphi_{l+1}, \cdots, \varphi_{l+m-1}$,
$E_1 = E_0^{\perp}$.

Let π_0 and π_1 be orthogonal projection onto E_0 and E_1, respectively. For any $K \in \mathcal{L}_+^2$, we define

$$A(K) = -\pi_0 K \pi_1 + \pi_1 K \pi_0$$

and

$$\Psi(K) = \exp A(K)(\pi_0 K \pi_0 + \pi_1 K \pi_1) \exp -A(K).$$

Then $\Psi : \mathcal{L}_+^2 \to \mathcal{L}_+^2$ is a C^∞ mapping and $\Psi(K^0) = K^0$. Implicit function theorem proves that there is two neighbourhoods $\mathfrak{U}(K^0)$ and $\mathfrak{V}(K^0)$ of K^0 in $\mathcal{L}_+^2$ such that Ψ is a diffeomorphism of $\mathfrak{U}(K^0)$ and $\mathfrak{V}(K^0)$. We can easily prove that

$$\begin{aligned}\mathfrak{V}(K^0) \cap Q_{lm} = \{K \in \mathfrak{V}(K^0) \mid (&\Psi^{-1}(K)\varphi_{l+p}, \varphi_{l+q}) \\ &= \delta_{pq}(\Psi^{-1}(K)\varphi_l, \varphi_l) \qquad 1 \le p, q \le m-1\}.\end{aligned}$$

Theorem 3 is proved.

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