20. A Note on the Covering Dimension of Lašnev Spaces

By Shinpei OKA
Shizuoka University

(Communicated by Kôsaku YOSIDA, M. J. A., March 13, 1978)

1. Introduction. In this note a space means a topological space with no separation axiom unless otherwise specified, and a map means a continuous map. A space X is called a Lašnev space if X is the closed image of a metric space, and $\dim X$ means the covering dimension of X.

Concerning the covering dimension of Lašnev spaces, I. M. Leĭbo [3] recently proved the following theorem.

Theorem 1. *If X is a Lašnev space with $\dim X \leqq n$, then there exist a normal space X_0 with $\dim X_0 \leqq 0$ and a closed map f from X_0 onto X with $\operatorname{ord} f \leqq n+1$, where $\operatorname{ord} f = \sup\{|f^{-1}(x)| : x \in X\}$.*

In this note we shall prove the following theorem asserting that in Theorem 1 we can replace 'normal space' by 'Lašnev space'. Our theorem seems to be interesting as a generalization of a theorem of K. Morita [4] for the case of X being metrizable.

Theorem 2. *A space X is a Lašnev space with $\dim X \leqq n$ if and only if there exist a Lašnev space X_0 with $\dim X_0 \leqq 0$ and a closed map f from X_0 onto X with $\operatorname{ord} f \leqq n+1$.*

The author would like to thank Prof. K. Morita for his advice which is useful to simplify our original proof of Theorem 2.

2. Preliminaries and lemmas. For a normal space X with $\operatorname{Ind} X = n$, a closed set F and its open nbd (=neighbourhood) G are said to determine $\operatorname{Ind} X$ if $\operatorname{Ind} Bd(U) \geqq n-1$ holds for every open nbd U of F with $\overline{U} \subset G$, where $\operatorname{Ind} X$ is the large inductive dimension of X.

In [3], the concepts of special families and special maps were introduced to prove that $\operatorname{Ind} X = \dim X$ for any Lašnev space X.

A countable family $\Phi = \{F_i, G_i\}_{i=1}^{\infty}$ of closed sets F_i and corresponding open nbds G_i in a normal space X is called a special family if for each closed set X' of X there exists a natural number j such that the pair $\{X' \cap F_j, X' \cap G_j\}$ determines $\operatorname{Ind} X'$.

A map g from a normal space X, provided with a special family $\Phi = \{F_i, G_i\}_{i=1}^{\infty}$, onto a metric space S is called a special map relative to Φ if it satisfies the following conditions.

(1) $g(F_i)$ is a closed set of S for every i.

(2) For each i, there exists an open nbd U_i of $g(F_i)$ in S such that $g^{-1}(U_i) \subset G_i$.

In particular g is called a special contraction if g is one-to-one.

The following diagram, called a pullback square, will play a key role in this note as well as in [3]. Hereafter every diagram consists of spaces and maps.

Commutative diagram 1 below is called a pullback square if, for any commutative diagram 2, there exists a unique map t' from T' to T which makes diagram 3 commutative.

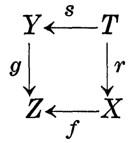

Diagram 1.

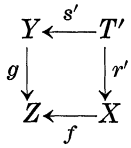

Diagram 2.

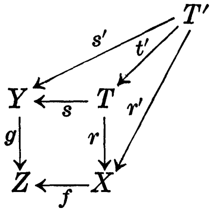

Diagram 3.

One can regard T as the space $\{(x, y) \in X \times Y : f(x) = g(y)\}$ with the relative topology in $X \times Y$, and r and s as the restrictions to T of the projections from $X \times Y$ to X and Y respectively.

In the following lemma, (1) is known and (2) is easily seen.

Lemma 3. *In the pullback square above:*

(1) *If f is a perfect map, then so is s.*

(2) *If* $\operatorname{ord} f \leqq n$ *for a positive integer n, then* $\operatorname{ord} s \leqq n$.

The following lemma will be used to prove Theorem 2.

Lemma 4. *In the pullback square above, r is a closed map under the conditions that:*

(1) *X is a Hausdorff space.*

(2) *f is a closed map with $|f^{-1}(z)| < \aleph_0$ for each $z \in Z$.*

(3) *g is a closed map.*

Proof. Let x be any point of Im (r) ($=$Image (r)) and D any open nbd of $r^{-1}(x)$ in T. Let us write $f^{-1}(f(x)) = \{x_0, x_1, \cdots, x_k\}$ where $x_0 = x$. Then we can take disjoint open sets U_0 and U_1 of X such that $x_0 \in U_0$ and $\{x_1, \cdots, x_k\} \subset U_1$. Let us put $E = (r^{-1}(U_0) \cap D) \cup r^{-1}(U_1)$. Then E is an open nbd of $(g \circ s)^{-1}(f(x_0))$ in T. Since, by (2), (3) and Lemma 3, $g \circ s$ is a closed map, we find an open nbd H of $f(x_0)$ in Z such that $(g \circ s)^{-1}(f(x_0)) \subset (g \circ s)^{-1}(H) \subset E$. Thus $U_0 \cap f^{-1}(H)$ is an open nbd of x_0 in X such that

$$r^{-1}(U_0 \cap f^{-1}(H)) = r^{-1}(U_0) \cap (g \circ s)^{-1}(H) \subset r^{-1}(U_0) \cap E$$
$$= r^{-1}(U_0) \cap D \subset D.$$

Hence r is a closed map onto Im (r). It is to be noticed that Im (r) is a closed set of X because Im $(r)=f^{-1}(\text{Im}\,(f)\cap \text{Im}\,(g))$. Thus we complete the proof of Lemma 4.

3. **Proof of Theorem 2.** The 'if'-part is well-known (cf. [5] 9.2.13). To show the 'only if'-part, let Y be a Lašnev space with dim $Y\leqq n$. Then there exist a special family Φ of Y, a metric space Z with dim $Z\leqq n$ and a special contraction g from Y onto Z relative to Φ ([2]). Furthermore there exist a metric space X with dim $X\leqq 0$ and a closed map f from X onto Z with ord $f\leqq n+1$ ([4]). Then we obtain the space T and the maps r and s such that the lower square in diag. 4 below is a pullback square. By Lemma 3, s is a closed map with ord s $\leqq n+1$, and T is clearly a perfectly normal paracompact Hausdorff space. It has been proved in [3] that Ind $T\leqq 0$ and hence dim $T\leqq 0$. (These arguments are due to [3] which, however, lacks the observation that T is just a Lašnev space as is shown in the following.) Since Y is a Lašnev space, there exist a metric space M and a closed map h from M onto Y. Then we obtain the space T' and the maps r' and s' such that the outer square in diag. 4 is a pullback square. By the definition of pullback squares, there exists a unique map t' from T' onto T which makes diag. 4 commutative. Then the upper square in diag. 4 is also a pullback square as is well-known (cf. [1] Exercise 21 E.). Hence, by Lemma 4, t' is a closed map from T' onto T. On the other hand, T' is a metric space as a subspace of a metric space $X\times M$. Consequently T is a Lašnev space, and the proof is completed.

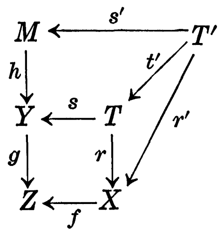

Diagram 4.

References

[1] H. Herrlich and G. E. Strecker: Category Theory. Allyn and Bacon Inc., Boston (1973).

[2] I. M. Leĭbo: On the equality of dimensions for closed images of metric spaces. Soviet. Math. Dokl., **15**, 835–839 (1974).

[3] ——: On closed images of metric spaces. ibid., **16**, 1292–1295 (1975).

[4] K. Morita: A condition for the metrizability of topological spaces and for n-dimensionality. Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A, **5**, 33–36 (1955).

[5] A. R. Pears: Dimension Theory of General Spaces. Cambridge University Press, Cambridge (1975).