2. Toda Brackets in the EHP Sequence

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1. Introduction and Theorems. In the computation of homotopy groups of spheres it is well-known that the formula $H\{E\alpha, E\beta, E\gamma\} = -\Delta^{-1}(\alpha\beta)E^2\gamma$ due to H. Toda [9] and its analogue for the tertiary bracket [8, 5] have played a dominant role. In this note we present two more formulae which, together with the Toda formula, give us a complete description of the behavior of Toda brackets in the EHP sequence

$$\cdots \longrightarrow \pi_i^n \xrightarrow{E} \pi_{i+1}^{n+1} \xrightarrow{H} \pi_{i+1}^{2n+1} \xrightarrow{\Delta} \pi_{i-1}^n \longrightarrow \cdots$$

where π_i^n denotes the 2 primary component of $\pi_i(S^n)$.

Theorem 1.1. *We have the relations, setting* $N_r = \ker (E^r\gamma)^*$,

(1) $E\{\Delta\alpha, \beta, \gamma\} = -H^{-1}(\alpha\circ E^2\beta)E^2\gamma$, $\quad E\{\Delta\alpha, \beta, \gamma, \delta\} \subset -\{H^{-1}(\alpha\circ E^2\beta) \cap N_2, E^2\gamma, E^2\delta\}$.

(2) $H\{E\alpha, E\beta, E\gamma\} = -\Delta^{-1}(\alpha\beta)E^2\gamma$, $\quad H\{E\alpha, E\beta, E\gamma, E\delta\} \subset -\{\Delta^{-1}(\alpha\beta) \cap N_2, E^2\gamma, E^2\delta\}$.

(3) $\Delta\{H\alpha, E\beta, E\gamma\} = -E^{-1}(\alpha\circ E\beta)\gamma$, $\quad \Delta\{H\alpha, E\beta, E\gamma, E\delta\} \subset -\{E^{-1}(\alpha\circ E\beta) \cap N_0, \gamma, \delta\}$.

I. M. James [2] has constructed, for $r \leqq 3n-2$, an exact sequence

$$\cdots \longrightarrow \pi_r^n \xrightarrow{E^k} \pi_{r+k}^{n+k} \xrightarrow{H_k} \pi_{r-n}(V_{n+k,k}) \xrightarrow{P_k} \pi_{r-1}^n \longrightarrow \cdots.$$

By the same principle as in the proof of Theorem 1.1 we deduce

Theorem 1.2. *Suppose* $\beta \in \pi_{s-n-1}^{r-n-1}$, $\gamma \in \pi_{t-n-1}^{s-n-1}$ *satisfy* $\beta\gamma = 0$, $t \leqq 3n-2$. *Then we have*

(1) $E^k\{P_k\alpha, E^n\beta, E^n\gamma\} = -H_k^{-1}(\alpha\circ E\beta)E^{n+k+1}\gamma$ *for* $\alpha \in \pi_{r-n}(V_{n+k,k})$ *with* $P_k(\alpha\circ E\beta) = 0$.

(2) $H_k\{E^k\alpha, E^{n+k+1}\beta, E^{n+k+1}\gamma\} = -P_k^{-1}(\alpha\circ E^{n+1}\beta)E^2\gamma$ *for* $\alpha \in \pi_r^n$ *with* $E^k(\alpha\circ E^{n+1}\beta) = 0$.

(3) $P_k\{H_k\alpha, E\beta, E\gamma\} = -E^{-k}(\alpha\circ E^{n+k+1}\beta)E^{n+1}\gamma$ *for* $\alpha \in \pi_{r+k}^{n+k}$ *with* $H_k(\alpha\circ E^{n+k+1}\beta) = 0$.

One of these formulae provides us with a useful tool for the problem of desuspending Whitehead products and gives us a constructive ground for some propositions proved in Toda [9], as shown in the following section.

The following theorem gives us a method for constructing an element from an element of even dimension, which may be proved by a theorem due to I. M. James [4].

Theorem 1.3. *Let $\partial : \pi_r^n \to \pi_{r-1}^{n-1}$ denote the transgression for the homotopy sequence of the fibration $p_{n+1,2} : V_{n+1,2} \to S^n$, where n is even. Let $t_{n+1,k+2} \in \pi_{n-1}(V_{n,k+1})$ denote the characteristic class for $p_{n+1,k+2} : V_{n+1,k+2} \to S^n$. Then*

(1) $E^2\partial(\tau) = 2E\tau$.

(2) *For $\tau \in \pi_r^n$ with $r \leqq 3n-2k-4$*

$$E\partial(\tau) = 2\tau + [EH_1(\tau), \iota_k],$$
$$i_{n,k+1*}H_k\partial(\tau) = -2H_{k+1}(\tau) + t_{n+1,k+2}H_1(\tau),$$

where $i_{n,k+1} : V_{n-1,k} \to V_{n,k+1}$ denotes the fibre inclusion. Further, if $p_{n,k+1}$ has a section $s_{n,k+1} : S^{n-1} \to V_{n,k+1}$ and if

$$H_{k+1}(\tau) = s_{n,k+1*}H_1(\tau) + i_{n,k+1*}(\tau'), \qquad \tau' \in \pi_{r-n}(V_{n-1,k})$$
$$t_{n+1,k+2} = 2s_{n,k+1*}\iota_{n-1} + i_{n,k+1*}(\mu), \qquad \mu \in \pi_{n-1}(V_{n-1,k})$$

then

$$H_k\partial(\tau) = \mu H_1(\tau) - 2\tau'.$$

2. Applications to Whitehead products. It is known from the works of J. F. Adams [1] and I. M. James [3] that, for $n = 2^c 16^d m$ with odd m and $0 \leqq c \leqq 3$, $p_{n,k} : V_{n,k} \to S^{n-1}$ has a section $s_{n,k}$ and that $H_{k-1}[\iota_{n-1}, \iota_{n-1}] = 0$ but $H_k[\iota_{n-1}, \iota_{n-1}] = -t_{n,k+1} \neq 0$, where $k = 2^c + 8d$. Let $\tau_{n-k}^{(k-1)} \in \pi_{2n-k-2}^{n-k}$ denote the element $P_k(-s_{n,k*}\iota_{n-1})$. Then we see that

$$E^{k-1}\tau_{n-k}^{(k-1)} = [\iota_{n-1}, \iota_{n-1}], \qquad H_1\tau_{n-k}^{(k-1)} \not\equiv 0 \bmod 2\pi_{n-2}^{n-k-1} \quad \text{if } n-k \text{ is even.}$$

Theorem 1.1, (1) implies

Theorem 2.1. *Suppose $\tau_{n-k}^{(k-1)}(E^{n-k}\beta) = 0$, $\beta\gamma = 0$, $H_1\alpha = E\beta$ for $\alpha \in \pi_{i+n+1}^n$, $\beta \in \pi_i^{n-2}$, $\gamma \in \pi_j^i$, $j \leqq 2n-6$. Then for any element $T_{k-1}(\beta, \gamma) \in \{\tau_{n-k}^{(k-1)}, E^{n-k}\beta, E^{n-k}\gamma\}$ we have*

$$(-1)^{k-1}E^kT_{k-1}(\beta, \gamma) \equiv \alpha(E^{n+1}\gamma) \bmod E\pi_{i+n}^{n-1}(E^{n+1}\gamma).$$

If $\pi_{i+n}^{n-1} = E^{k-1}\pi_{i+n-k+1}^{n-k}$ then $T_{k-1}(\beta, \gamma)$ may be chosen so that $(-1)^{k-1}E^k T_{k-1}(\beta, \gamma) = \alpha(E^{n+1}\gamma)$.

Theorem 1.2, (1) yields

Theorem 2.2. *Suppose that $\alpha = E^{k-1}\tau$, $H_1\tau = E\beta$, $\beta\gamma = 0$ for $\alpha \in \pi_{n+i}^n$, $\tau \in \pi_{n+i-k+1}^{n-k+1}$, $\beta \in \pi_{i-1}^{n-k-1}$, $\gamma \in \pi_j^{i-1}$ with $j \leqq 2n-2k-3$. Then there exists an element $S_k(\beta, \gamma) \in \{[\iota_{n-k}, \iota_{n-k}], E^{n-k}\beta, E^{n-k}\gamma\}$ such that*

$$(-1)^{k-1}E^kS_k(\beta, \gamma) = \alpha(E^{n+1}\gamma).$$

Further if $\gamma\delta = 0$ for $\delta \in \pi_q^j$ then $S_k(\beta, \gamma)E^{n-k+1}\delta \in -P_1\{E^{n-k}\beta, E^{n-k}\gamma, E^{n-k}\delta\}$. In particular, for $\alpha = [\theta, \iota_n]$, $\theta \in \pi_{i+1}^n$, we have

$$(-1)^{k-1}E^kS_k(\beta, \gamma) = [\theta(E^2\gamma), \iota_n],$$
$$i_{n,k+1*}^{(k)}H_1S_k(\beta, \gamma) = t_{n+1,k+2}(E^{-1}\theta)E^n\gamma.$$

From Theorem 1.2, (3) one can deduce

Theorem 2.3. *Let $[E^2\beta, \iota_n] = E^k\tau$, $\beta\gamma = 0$ for $\beta \in \pi_i^{n-2}$, $\tau \in \pi_{i+n-k+1}^{n-k}$, $\gamma \in \pi_j^i$ with $j \leqq 3n-3k-4$. Then*

$$\tau(E^{n-k+1}\gamma) \in P_k\{t_{n+1,k+1}, E\beta, E\gamma\}.$$

These theorems enable us to construct meta-stable elements whose iterated suspensions are Whitehead products and among which we may find the elements obtained by N. Oda [7]. We list below several consequences of the above theorems.

1) For odd n, $ET_0(\beta, 2\iota_i)=2\alpha$ and $H_1T_0(\beta, 2\iota_i)\equiv\beta\eta_i$ where $\alpha\in\pi^n_{i+n+1}$, $\beta\in\pi^{n-2}_i$, $2\beta=0$, $H_1\alpha=E\beta$, $i\leqq 2n-6$.

2) For $n\equiv 3 \bmod 4$, $ET_0(\eta^2_{n-2},\gamma)=(\partial\tau^{(1)}_{n+1})E^{n+1}\gamma$, where $\gamma\in\pi^n_j$, $\eta^2_{n-2}\gamma=0$, $j\leqq 2n-6$. We may take ε_n, $\eta_n\rho_{n+1}$, κ_n, ϕ_n for γ.

3) For $n\equiv 2 \bmod 4$, $E^2T_1(2\iota_{n-2},\gamma)=[E^2\gamma,\iota_n]$, $H_1T_1(2\iota_{n-2},\gamma)\in\{\eta_{n-3}, 2\iota_{n-2},\gamma\}$ where $\gamma\in\pi^{n-2}_j$, $2\gamma=0$, $j\leqq 2n-6$. We may take ν^2_{n-2}, σ^2_{n-2}, $8\sigma_{n-2}$, $16\rho_{n-2}$, $\bar{\sigma}_{n-2}$ for γ.

4) For $n\equiv 4 \bmod 8$, $E^4T_3(8\iota_{n-2},\gamma)=[4E^2\gamma,\iota_n]$, $H_1T_3(8\iota_{n-2},\gamma)\in\{\nu_{n-5}, 8\iota_{n-2},\gamma\}$ where $\gamma\in\pi^{n-2}_j$, $8\gamma=0$, $j\leqq 2n-6$. We may take ν_{n-2}, $2\sigma_{n-2}$, $4\rho_{n-2}$, ζ_{n-2}, $\bar{\zeta}_{n-2}$ for γ.

5) For $n\equiv 4 \bmod 8$, $E^4T_3(\eta_{n-2},\gamma)=\tau^{(1)}_nE^{n+1}\gamma$, $H_1T_3(\eta_{n-2},\gamma)\in\{\nu_{n-5},\eta_{n-2},\gamma\}$, where $\gamma\in\pi^{n-1}_j$, $\eta_{n-2}\gamma=0$, $j\leqq 2n-6$. We may take ν_{n-1}, σ^2_{n-1}, $\eta_{n-1}\varepsilon_n$, $\eta_{n-1}\kappa_n$, ζ_{n-1} for γ.

6) For $n\equiv 6 \bmod 8$, $E^2T_1(\nu^2_{n-2},\gamma)=\bar{\tau}(E^{n+1}\gamma)$, $H_1T_1(\nu^2_{n-2},\gamma)\in\{\eta_{n-3},\nu^2_{n-2},\gamma\}$ where $\gamma\in\pi^{n+4}_j$, $\nu^2_{n-2}\gamma=0$, $j\leqq 2n-6$ and $\bar{\tau}_n\in\pi^n_{2n+5}$ is an element given in Y. Nomura [6]. We may take $2\iota_{n+4}$, ν^2_{n+4}, ν^*_{n+4}, $\bar{\sigma}_{n+4}$ for γ.

7) For $n\equiv 4 \bmod 8$, $E^4T_3(2\nu_{n-2},\gamma)=[\nu_n(E^2\gamma),\iota_n]$, $H_1T_3(2\nu_{n-2},\gamma)\in\{\nu_{n-5}, 2\nu_{n-2},\gamma\}$ where $\gamma\in\pi^{n+1}_j$, $2\nu_{n-2}\gamma=0$, $j\leqq 2n-6$. We may take $4\iota_{n+1}$, κ_{n+1}, ν^*_{n-1}, η_{n+1}, $4\bar{\kappa}_{n+1}$ for γ.

8) For $n\equiv 8 \bmod 16$ with $[8\sigma_{n-8},\iota_{n-8}]\neq 0$, $E^8T_7(32\iota_{n-2},\gamma)=[16E^2\gamma,\iota_n]$, $H_1T_7(32\iota_{n-2},\gamma)\in\{\sigma_{n-9}, 32\iota_{n-2},\gamma\}$ where $\gamma\in\pi^{n-2}_j$, $32\gamma=0$, $j\leqq 2n-6$. We may take ρ_{n-2}, $2\rho_{3,n-2}$ for γ.

9) For $n\equiv 4 \bmod 8$ there exists an element $\bar{\nu}_{n-4}\in\{\tau^{(3)}_{n-4},\eta_{2n-6}, 2\iota_{2n-5},\bar{\nu}_{2n-5}\}$ such that $2E^4\bar{\nu}_{n-4}=\tau^{(1)}_n\nu^3_{2n}$, $E^6\bar{\nu}_{n-4}=[\bar{\nu}_{n+2},\iota_{n+2}]$.

10) For $n\equiv 4 \bmod 8$ there exists an element $\bar{\kappa}_{n-4}\in\{\tau^{(3)}_{n-4},\eta_{2n-6}, 2\iota_{2n-5},\kappa_{2n-5}\}$ such that $4E^3\bar{\kappa}_{n-4}=[\nu_{n-1}\kappa_{n+2},\iota_{n-1}]$, $2E^4\bar{\kappa}_{n-4}=\tau^{(1)}_n\eta_{2n}\kappa_{2n+1}$, $E^6\bar{\kappa}_{n-4}=[\kappa_{n+2},\iota_{n+2}]$.

11) For $n\equiv 0 \bmod 4$, $\tau\in\pi^n_r$, $r\leqq 3n-6$ we have $H_1\partial(\tau)=\eta_{n-2}H_1\tau$.

12) For $\tau\in\pi^n_r$, $n\equiv 2 \bmod 4$, $2H_1\tau=0$, $\eta_{n-3}E^{-1}H_1\tau=0$, $r\leqq 3n-6$ there exists an element τ_1, $\tau_2\in\pi^{n-2}_{r-2}$ such that $E\tau_1=\partial(\tau)$, $E^2\tau_2=[EH_1\tau,\iota_n]$, $H_1(\tau_1-\tau_2)\in\{\eta_{n-3}, E^{-1}H_1\tau, 2\iota_{r-n-1}\}$.

13) For $n\equiv 0 \bmod 8$ there exist $\lambda_{n-3}\in\pi^{n-3}_{2n+1}$ of order 8, λ'_{n-5} and $\lambda''_{n-6}=T_0(\alpha, 2\iota_{n+1})$ ($\alpha=\varepsilon_{n-7}$ or $\bar{\nu}_{n-7}$) such that $E^2\lambda_{n-3}=\partial\tau^{(3)}_n$, $E^5\lambda_{n-3}=2E^2\tau^{(3)}_n=[\eta_{n+2},\iota_{n+2}]$, $2E^4\lambda_{n-3}=[\eta^2_{n+1},\iota_{n+1}]$, $E^2\lambda'_{n-5}=2\lambda_{n-3}$, $E\lambda''_{n-6}=2\lambda'_{n-5}$, $H_1\lambda_{n-3}=\nu^2_{n-4}$, $H_1\lambda'_{n-5}\equiv\varepsilon_{n-6}$, $H_1\lambda''_{n-6}\equiv\eta_{n-7}\varepsilon_{n-6}$. Further there exists an element $\bar{\lambda}_{n-8}\in\pi^{n-8}_{2n-3}$ such that $E^2\bar{\lambda}_{n-8}=S_3(E^{-1}H_1\lambda'_{n-5},\nu_{n+1})$, $E^5\bar{\lambda}_{n-8}=2\lambda_{n-3}\nu_{2n-1}$, $H_1\bar{\lambda}_{n-8}=\kappa_{n-9}$, $E^7\bar{\lambda}_{n-8}=0$.

14) Let $\nu_{n-2}\gamma=0$ for $\gamma\in\pi^{n+1}_j$, $n\equiv 8 \bmod 16$, $j\leqq 2n-2$. Then, for any element $\delta\in\{E^4\tau^{(7)}_{n-8},\nu_{2n-6}, E^{n-4}\gamma\}$, we have $E^4\delta\equiv\tau^{(3)}_n(E^{n+1}\gamma) \bmod 2\tau^{(3)}_n(E^{n+1}\gamma)$, hence $E^7\delta=[E^2\gamma,\iota_{n+3}]$.

15) For $n\equiv 0 \bmod 4$, $\gamma\in\pi_j^{n-2}$, $j\leqq 2n-5$, $2\gamma=0$, we have $ES_1(2\iota_{n-2},\gamma)=[E^2\gamma,\iota_n]$, $H_1S_1(2\iota_{n-2},\gamma)=\eta_{n-2}(E\gamma)$.

16) We have relations: $\bar{\tau}_{n-3}\gamma_{2n-1}=0$ for $\gamma=\eta,\ \varepsilon,\ \bar{\nu},\ \mu,\ \sigma$; $(\partial\tau_n^{(1)})\gamma_{2n-1}=0$ for $\gamma=\nu,\bar{\sigma},\sigma^2,\eta\varepsilon,\eta^2\rho,\eta\kappa,\zeta,\bar{\zeta}$; $(\partial\tau_n^{(7)})\ \gamma_{2n+5}=0$ for $\gamma=\eta^3,\eta\sigma$.

The details of our results will appear elsewhere.

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