83. Completeness Criterion for Functions with Delay Defined over a Domain of Three Elements

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§1. **Introduction.** Let $E_k=\{0,1,\cdots,k-1\}$ for $k\geq 2$. A *k-valued function with delay* is defined to be a pair (f,d), where f is a usual k-valued logical (or switching) function, that is, a function from a cartesian product $E_k\times E_k\times\cdots\times E_k$ into E_k, and d is a nonnegative integer. It represents a certain switching element which performs its operation in d units of time delay. A set of such functions with delay is called *complete* if any k-valued function can be realized with some delay by composing the elements in the set "synchronously", i.e., so as to synchronize the delays caused at each part of the composition. A *spectrum* over E_k is an infinite sequence of sets of k-valued functions indexed by nonnegative integers, and it is used as an algebraic representation of a set of functions with delay. A spectrum is complete if and only if it is not included in any *maximal incomplete* spectrum, so that explicit determination of maximal spectra is important in order to effectively decide the completeness of an arbitrary given set of functions with delay. The above notions are developed as an extension of the classical theory on completeness for k-valued functions (e.g., [2], [5] and [6]).

V. B. Kudrjavcev [3] first considered the above completeness problem in the binary case $k=2$ (he called *completeness in the second sense*), and gave a criterion for completeness by explicitly determining all 11 classes of maximal spectra over E_2. For the case of general k, A. Nozaki [4] gave a criterion for completeness, and T. Hikita and A. Nozaki [1] gave a characterization of maximal spectra.

In this note, after introducing some preliminary results (§2), we reformulate the characterization of maximal spectra in [1] by using the notion of relation and classify them into three types named (A), (B) and (C) (§3). Then we apply this characterization to the three-valued case, and report the result of the explicit determination of all maximal spectra over E_3 (§4). The method is essentially in the same spirit as that in [6], and it is based on several lemmas on inclusion

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among spectra of these types.

Thus the completeness problem is solved to the final form in the case $k=3$. It turns out that there exist 18 maximal spectra of type (A), 19 of type (C), and 12 classes of infinitely many maximal spectra of type (B); in total, 49 distinct classes.

§2. Functions, relations and spectra. Let P_k^n denote the set of all functions from the cartesian product $(E_k)^n$ $(n\geq 1)$ into E_k, and let P_k denote the set $\bigcup_{n=1}^{\infty} P_k^n$. The notions of *closure*, *closedness*, *completeness* and *maximal incompleteness* are defined for a set of functions in P_k as usual.

For $h\geq 1$, the cartesian product $(E_k)^h$ is denoted by π_h. A subset of π_h is called an h-ary *relation* on E_k ([6]). The set of all ordered h-tuples in π_h where elements are pairwise distinct, is denoted by σ_h, and we put $\iota_h=\pi_h-\sigma_h$. For example, $\iota_2=\{(a,a)\,;\,a\in E_k\}$. An h-ary relation Δ on E_k is called *diagonal* ([6]) if there exists a direct sum decomposition $\bigcup_m \mathcal{E}_m$ of the set $\{0,1,\cdots,h-1\}$ such that

$$\Delta=\{(a_0,a_1,\cdots,a_{h-1})\,;\,\text{if } i,j\in\mathcal{E}_m \text{ then } a_i=a_j\}.$$

For a function f and an h-ary relation ρ, an h-ary relation $f(\rho)$ is defined by

$$f(\rho)=\{(f(a_0^0,a_0^1,\cdots,a_0^{n-1}),\cdots,f(a_{h-1}^0,a_{h-1}^1,\cdots,a_{h-1}^{n-1}))\,;\,(a_0^0,a_1^0,\cdots,a_{h-1}^0),\cdots,(a_0^{n-1},a_1^{n-1},\cdots,a_{h-1}^{n-1}) \text{ are in } \rho\}.$$

Definition. For two h-ary relations ρ_0 and ρ_1 on E_k, $A(\rho_0,\rho_1)$ is a set of functions defined by $A(\rho_0,\rho_1)=\{f\in P_k\,;\,f(\rho_0)\subset\rho_1\}$.

For every maximal set M in P_k there exist some h-ary relations ρ on E_k such that $M=A(\rho,\rho)$ ([6]). We denote one of these ρ by $\alpha[M]$. Actually we can choose h between 1 and k, except for the set of all "linear" functions in P_2.

A *spectrum* over E_k is an infinite sequence $\mathcal{F}=(F_d)_{d\geq 0}=(F_0,F_1,\cdots,F_d,\cdots)$ of subsets of P_k, and each F_d represents the set of functions with delay d. Such notions as *closure*, *closedness*, *completeness* and *maximal incompleteness* for a spectrum are defined analogously to those for a set of functions (see [1], in which they are called ~-closure, etc.). The following is the basis of this paper ([1], [3], and Corrigenda to [1] (to appear)).

Theorem 2.1. *A spectrum over E_k is complete if and only if it is not included in any maximal spectrum over E_k.*

§3. Characterization of maximal spectra. The characterization of maximal spectra in [1] can be reformulated in terms of relations.

Definition. A spectrum $\mathcal{F}=(F_d)_{d\geq 0}$ over E_k is called *type* (A) if there exists a maximal set M in P_k such that $F_d=M$ for all $d\geq 0$.

Definition. An ordered p-tuple $(p\geq 1)$ of h-ary relations on E_k $\bar{\rho}=(\rho^0,\rho^1,\cdots,\rho^{p-1})$ is called an h-ary *polyrelation* on E_k with *period* p.

Definition. For a polyrelation $\bar{\rho}=(\rho^0,\rho^1,\cdots,\rho^{p-1})$ with period p, a spectrum $\mathcal{B}(\bar{\rho})=(F_d)_{d\geq 0}$ is defined by $F_d=\bigcap_{m=0}^{p-1} A(\rho^m, \rho^{m\oplus d})$ for all $d\geq 0$, where $\oplus$ denotes the addition modulo p. A spectrum $\mathcal{F}$ is called *type* (B) if there exists a polyrelation $\bar{\rho}$ such that $\mathcal{F}=\mathcal{B}(\bar{\rho})$.

Definition. For two h-ary relations ρ and Δ where Δ is diagonal, a spectrum $\mathcal{C}(\rho,\Delta)=(F_d)_{d\geq 0}$ is defined by $F_0=A(\rho,\rho)$ and $F_d=A(\rho,\Delta)$ for all $d\geq 1$. A spectrum $\mathcal{F}$ is called *type* (C) if there exist relations ρ and Δ such that $\mathcal{F}=\mathcal{C}(\rho,\Delta)$.

Note that for each fixed value of k there exist only a finite number of spectra of types (A) and (C), but an infinite number of those of type (B).

Proposition 3.1. *A spectrum of type* (A) *is maximal.*

Proposition 3.2. *Let* $\bar{\rho}=(\rho^0, \rho^1, \cdots, \rho^{p-1})$ *be a polyrelation. Then* $\mathcal{B}(\bar{\rho})$ *is closed. It is complete if and only if every* ρ^m, $m=0,1,\cdots, p-1$, *is diagonal or* $\emptyset$.

Proposition 3.3. *Let* ρ *and* Δ *be h-ary relations and* Δ *be diagonal. Then* $\mathcal{C}(\rho,\Delta)$ *is closed. It is complete, if and only if,* $\rho\subset\Delta$, *or,* ρ *is diagonal or* $\emptyset$.

The following characterization is immediate from that in [1]:

Theorem 3.4. *A maximal spectrum over* E_k *is either of type* (A), *type* (B) *defined by an h-ary polyrelation with* $1\leq h\leq k$ *and period* $p\geq 2$, *or type* (C) *defined by h-ary relations with* $1\leq h\leq k$.

§4. Maximal spectra over E_3. S. V. Jablonskii [2] determined all 18 maximal sets in P_3. Thus we have:

Theorem 4.1. *A maximal spectrum of type* (A) *over* E_3 *is the one all of whose components are equal to a maximal set in* P_3 *listed in* 1)–18).

In the following a maximal set is shown by the relation which defines the set and by the name given in [2].

1) $\{(0)\}$; $T^3_{\mathcal{E}_0,0}$ 2) $\{(1)\}$; $T^3_{\mathcal{E}_1,0}$ 3) $\{(2)\}$; $T^3_{\mathcal{E}_2,0}$
4) $\{(0),(1)\}$; $T^3_{\mathcal{E}_{01},0}$ 5) $\{(0),(2)\}$; $T^3_{\mathcal{E}_{02},0}$ 6) $\{(1),(2)\}$; $T^3_{\mathcal{E}_{12},0}$
7) $\{(0,0),(1,1),(2,2),(0,1),(1,0),(0,2),(2,0)\}$; $T^3_{\mathcal{E}_0,1}$
8) $\{(0,0),(1,1),(2,2),(1,0),(0,1),(1,2),(2,1)\}$; $T^3_{\mathcal{E}_1,1}$
9) $\{(0,0),(1,1),(2,2),(2,0),(0,2),(2,1),(1,2)\}$; $T^3_{\mathcal{E}_2,1}$
10) $\{(0,0),(1,1),(2,2),(1,2),(2,1)\}$; $U^3_{\mathcal{E}_{12},\mathcal{E}_0}$
11) $\{(0,0),(1,1),(2,2),(0,2),(2,0)\}$; $U^3_{\mathcal{E}_{02},\mathcal{E}_1}$
12) $\{(0,0),(1,1),(2,2),(0,1),(1,0)\}$; $U^3_{\mathcal{E}_{01},\mathcal{E}_2}$
13) $\{(0,0),(1,1),(2,2),(0,1),(0,2),(1,2)\}$; M^3_1
14) $\{(0,0),(1,1),(2,2),(1,2),(1,0),(2,0)\}$; M^3_2
15) $\{(0,0),(1,1),(2,2),(2,0),(2,1),(0,1)\}$; M^3_3
16) $\{(0,1),(1,2),(2,0)\}$; S^3_{x+1}
17) $\{(a,b,c)\in\pi_3$; at least two of a,b,c are equal$\}$; $T^3_{N,2}$
18) $\{(a,b,c)\in\pi_3$; $a=b=c$, or, a,b,c are pairwise distinct$\}$; L^3

For a binary relation ρ, the *converse* relation ρ^c is defined by

$\rho^c=\{(b,a);(a,b) \text{ in } \rho\}$.

Theorem 4.2. *A spectrum of type* (B) *over* E_3 *is maximal if and only if it is equal to* $\mathcal{B}(\bar{\rho})$ *where* $\bar{\rho}$ *is one of the polyrelations listed in* 19)–30).

Let period $p=2^r$ for some $r\geq 1$, put $s=2^{r-1}$, and let $t\neq 0$, s.

	ρ^0	ρ^s	ρ^t	
19)	$\{(0)\}$	$\{(1)\}$	$\emptyset$	
20)	$\{(0)\}$	$\{(2)\}$	$\emptyset$	
21)	$\{(1)\}$	$\{(2)\}$	$\emptyset$	
22)	$\{(0)\}$	$\{(1),(2)\}$	$\emptyset$	
23)	$\{(1)\}$	$\{(0),(2)\}$	$\emptyset$	
24)	$\{(2)\}$	$\{(0),(1)\}$	$\emptyset$	
25)	$\alpha[M_1^3]$	$(\alpha[M_1^3])^c$	ι_2	$(\alpha[M_1^3)$ in 13))
26)	$\alpha[M_2^3]$	$(\alpha[M_2^3])^c$	ι_2	$(\alpha[M_2^3]$ in 14))
27)	$\alpha[M_3^3]$	$(\alpha[M_3^3])^c$	ι_2	$(\alpha[M_3^3]$ in 15))
28)	$\alpha[S_{x+1}^3]$	$(\alpha[S_{x+1}^3])^c$	$\emptyset$	$(\alpha[S_{x+1}^3]$ in 16))

Let period $p=3^r$ for some $r\geq 1$, put $s=3^{r-1}$, and let $t\neq 0$, $s, 2s$.

	ρ^0	ρ^s	ρ^{2s}	ρ^t
29)	$\{(0)\}$	$\{(1)\}$	$\{(2)\}$	$\emptyset$
30)	$\{(0)\}$	$\{(2)\}$	$\{(1)\}$	$\emptyset$

Example. The maximal spectrum $(F_d)_{d\geq 0}$ defined by the polyrelation 19) is:

$$F_{s(2q)}=\{f\in P_3; f(0,0,\cdots,0)=0 \text{ and } f(1,1,\cdots,1)=1\},$$

and

$$F_{s(2q+1)}=\{f\in P_3; f(0,0,\cdots,0)=1 \text{ and } f(1,1,\cdots,1)=0\}$$

for all $q\geq 0$, and $F_d=\emptyset$ for all d not multiple of s.

Theorem 4.3. *A spectrum of type* (C) *over* E_3 *is maximal if and only if it is equal to* $\mathcal{C}(\rho,\iota_2)$ *where* ρ *is one of the binary relations listed in* 31)–49).

31) $\{(0,0),(0,1),(0,2)\}$
32) $\{(1,0),(1,1),(1,2)\}$
33) $\{(2,0),(2,1),(2,2)\}$
34) $\{(0,0),(1,1),(0,1),(1,0)\}$
35) $\{(0,0),(2,2),(0,2),(2,0)\}$
36) $\{(1,1),(2,2),(1,2),(2,1)\}$
37) $\{(0,1)\}$ 38) $\{(0,2)\}$ 39) $\{(1,2)\}$
40) $\{(0,1),(1,0)\}$ 41) $\{(0,2),(2,0)\}$ 42) $\{(1,2),(2,1)\}$
43) $\{(0,1),(2,2)\}$ 44) $\{(0,2),(1,1)\}$ 45) $\{(1,2),(0,0)\}$
46) $\{(0,1),(1,0),(2,2)\}$
47) $\{(0,2),(2,0),(1,1)\}$
48) $\{(1,2),(2,1),(0,0)\}$
49) $\{(0,1),(1,2),(2,0)\}$

Example. The maximal spectrum $(F_d)_{d\geq 0}$ defined by the relation

31) is:

$$F_0 = T^3_{\mathcal{E}_0, 0} = \{f \in P_3 ; f(0, 0, \cdots, 0) = 0\},$$

and

$$F_d = \{f \in P_3 ; f \text{ is constant-valued}\}$$

for all $d \geq 1$.

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