81. On the Rate of Convergence of the Difference Finite Element Approximation for Parabolic Equations

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1. Introduction. Our purpose is to make some error analysis on the difference finite element approximation for the parabolic equation.

Let Ω be a bounded domain in R^2 whose boundary $\partial\Omega$ is smooth, and let $-\mathcal{A}$ be a uniformly elliptic differential operator of second order with smooth coefficients:

$$(1.1)\qquad -\mathcal{A}=\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}a_{ij}(t,x)\frac{\partial}{\partial x_j}+\sum_{j=1}^{2}b_j(t,x)\frac{\partial}{\partial x_j}+c(t,x).$$

We consider the following parabolic equation

$$(1.2)\qquad \frac{\partial u}{\partial t}+\mathcal{A}u=0 \qquad (0<t\leqslant T,\ x\in\Omega)$$

with the boundary condition

$$(1.3)\qquad u=0 \qquad (0<t\leqslant T,\ x\in\partial\Omega)$$

and with an initial condition

$$(1.4)\qquad u|_{t=0}=\varphi(x) \qquad (x\in\Omega).$$

Assuming $\varphi\in X=L^2(\Omega)$, we can reduce the equation (1.2) with (1.3) and (1.4) to the following evolution equation

$$(1.5)\qquad \frac{du}{dt}+A(t)u=0 \qquad (0<t\leqslant T)$$

with

$$(1.6)\qquad u(0)=\varphi.$$

in X. Here the operator $A(t)$ is the m-sectorial operator associated with the following sesqui-linear form $a_t(\ ,\ )$ on $V\times V$, where $V=H_0^1(\Omega)$:

$$(1.7)\qquad a_t(u,v)=\sum_{i,j=1}^{2}\int_\Omega a_{ij}(t,x)\frac{\partial}{\partial x_j}u\cdot\overline{\frac{\partial}{\partial x_i}v}dx-\sum_{j=1}^{2}\int_\Omega b_j(t,x)\frac{\partial}{\partial x_j}u\cdot\bar{v}dx$$
$$-\int_\Omega c(t,x)u\cdot\bar{v}dx \qquad (u,v\in V).$$

In order to discretize the equation (1.5) with (1.6), we introduce an approximate space V_h for each $h>0$ by triangulating Ω regularly with the size parameter h and adopting piecewise linear trial functions in the usual manner. As for precise definition of V_h, particularly in the case of curved boundaries, see Zlámal [7]. Here we note only that V_h satisfies the following three conditions:

i) V_h is of finite dimensional.

ii) $V_h \subset V$.

iii) The estimate

$$\inf_{\chi \in V_h} \|\chi - v\|_1 \leqslant Ch \|v\|_2 \qquad (v \in H^2(\Omega) \cap V)$$

holds true.

Here and hereafter the symbol C stands for a generic positive constant. Firstly we consider the equation

$$\frac{du_h}{dt} + A_h(t)u_h = 0 \qquad (0 < t \leqslant T) \tag{1.8}$$

with

$$u_h(0) = P_h\varphi \tag{1.9}$$

in V_h, as a semi-discretization of the equation (1.5) with (1.6). Here the operator $A_h(t): V_h \to V_h$ is the m-sectorial operator associated with $a_t|_{V_h \times V_h}$ through the identity:

$$a_t(u, v) = (A_h(t)u, v) \qquad (u, v \in V_h), \tag{1.10}$$

where (,) is the L^2-inner product. P_h is the orthogonal projection in X.

Furthermore we discretize the time variable as well as the space variables. We consider the equation

$$u_h^\tau(t+\tau) - u_h^\tau(t) + \tau A_h(t+\tau)u_h^\tau(t+\tau) = 0 \qquad (t = n\tau,\ n = 0, 1, 2, \cdots) \tag{1.11}$$

with

$$u_h^\tau(0) = P_h\varphi \tag{1.12}$$

for a small positive parameter τ. The solution $u_h^\tau = u_h^\tau(t)$ of the equation (1.11) with (1.12) is called the backward difference finite element approximation of the solution $u = u(t)$ of the equation (1.5) with (1.6).

We want to estimate the error $u_h^\tau(t) - u(t)$ in L^2-norm. To this end we estimate $\|u_h^\tau(t) - u_h(t)\|_0$ and $\|u_h(t) - u(t)\|_0$, while the estimate

$$\|u_h(t) - u(t)\|_0 \leqslant Ch^2/t \, \|\varphi\|_0 \qquad (0 < t \leqslant T) \tag{1.13}$$

is already known. See, Fujita-Suzuki [2] and Suzuki [5], [6]. See also Fujita-Mizutani [1] and Helfrich [3], for more restricted results.

In this paper, the estimate

$$\|u_h^\tau(t) - u_h(t)\|_0 \leqslant C_\gamma (\tau/t)^{1-\gamma} \|\varphi\|_0 \qquad (t = n\tau)\ (0 < \gamma \leqslant 1) \tag{1.14}$$

is derived with a constant $C_\gamma > 0$ depending on γ, which gives our final estimate

$$\|u_h^\tau(t) - u(t)\|_0 \leqslant C_\gamma (h^2/t + (\tau/t)^{1-\gamma}) \|\varphi\|_0 \qquad (t = n\tau)\ (0 < \gamma \leqslant 1). \tag{1.15}$$

2. Theorems on generation and approximation of evolution operators. The following theorem is due to Kato-Tanabe [4].

Theorem 1. *Suppose that the operator $A(t)$ $(0 \leqslant t \leqslant T)$ in X, a Banach space, satisfies the following four conditions.*

(A0) *$A(t)$ is a densely defined closed linear operator on X whose resolvent set $\rho(A(t))$ contains the set G:*

$$G = \{\lambda \in \boldsymbol{C};\ |\arg \lambda| \geqslant \theta_1\} \cup \{0\} \qquad (0 < \theta_1 < \pi/2),$$

with the inequality

(2.1) $$\|(\lambda-A(t))^{-1}\|\leqslant M/(|\lambda|+1) \qquad (\lambda\in G,\ t\in[0,T]).$$

(A1) *$A(t)^{-1}$ is continuously differentiable in t with respect to the operator norm in X.*

(A2) *The inequality*

(2.2) $$\left\|\frac{d}{dt}A(t)^{-1}-\frac{d}{ds}A(s)^{-1}\right\|\leqslant K\,|t-s|^{\alpha} \qquad (t,s\in[0,T])$$

holds with $K>0$ and α in $0<\alpha\leqslant 1$.

(A3) *The inequality*

(2.3) $$\left\|\frac{\partial}{\partial t}(\lambda-A(t))^{-1}\right\|\leqslant N/|\lambda|^{\rho} \qquad (\lambda\in G,\ t\in[0,T])$$

holds with $N>0$ and ρ in $0<\rho\leqslant 1$.

Then $A(t)$ generates a family of evolution operators: $X\to X$ of C^1-class which is denoted by $\{U(t,s)\}$.

We can make use of Theorem 1 not only to construct $u_h(t)$, the solution of (1.8) and (1.9), but also to derive the estimate for $u_h(t)$ uniform in h. That is, we have the following

Theorem 2. *Let X, V be a couple of Hilbert spaces with continuous inclusion $V\hookrightarrow X$ and $\{V_h\}$* be *a family of finite dimensional spaces contained in V. And let $a_t(\ ,\ ): V\times V\to C$ be a given sesqui-linear form satisfying the following inequalities with constants $C>0$ and $\delta>0$:*

(2.4) $$|a_t(u,v)|\leqslant C\,\|u\|_V\cdot\|v\|_V \qquad (u,v\in V)$$

(2.5) $$\operatorname{Re} a_t(u,u)\geqslant \delta\,\|u\|_V^2 \qquad (u\in V).$$

Suppose that another sesqui-linear form $\dot{a}_t(\ ,\ ): V\times V\to \boldsymbol{C}$ exists and satisfies the following inequalities:

(2.6) $$|\dot{a}_t(u,v)|\leqslant C\,\|u\|_V\cdot\|v\|_V \qquad (u,v\in V)$$

(2.7) $$|\dot{a}_t(u,v)-\dot{a}_s(u,v)|\leqslant C\,|t-s|\,\|u\|_V\cdot\|v\|_V \qquad (u,v\in V)$$

and the following equality:

(2.8) $$\lim_{t\to s}\ \sup_{\substack{u,v\in V\\ \|u\|_V,\ \|v\|_V\leqslant 1}}\left|\frac{a_t(u,v)-a_s(u,v)}{t-s}-\dot{a}_s(u,v)\right|=0.$$

Then the m-sectorial operator $A_h(t): V_h\to V_h$ associated with the sesqui-linear form $a_t|_{V_h\times V_h}$ satisfies the conditions (A0)–(A3) *with $\alpha=\rho=1$. Furthermore, the constants θ_1, M, K and N in these conditions depend only on the constants C and δ in* (2.4)–(2.7).

By virtue of Theorems 1 and 2, we obtain

(2.9) $$u_h(t)=U_h(t,0)P_h\varphi,$$

where $\{U_h(t,s)\}$ is the family of evolution operators generated by $A_h(t)$. Indeed we can construct the form $\dot{a}_t(\ ,\ )$ satisfying the relations (2.6)–(2.8) by differentiating a_{ij}, b_j and c in t in the right hand side of (1.7).

On the other hand, we see

(2.10) $u_h^\tau(n\tau)=(1+\tau A_h(n\tau))^{-1}(1+\tau A_h((n-1)\tau)^{-1}\cdots(1+\tau A_h(\tau))^{-1}P_h\varphi.$

With the aid of Theorem 2, following Theorem 3 yields our main results, (1.14) and (1.15), when applied for $A=A_h$ and $U=U_h$.

Theorem 3. *Under the conditions* (A0)–(A3) *with* $\alpha=\rho=1$, *the estimate*

(2.11) $$\|U(n\tau,0)-(1+\tau A(n\tau))^{-1}(1+\tau A((n-1)\tau))^{-1}\cdots(1+\tau A(\tau))^{-1}\| \leqslant C_\gamma 1/n^\gamma$$

holds for each γ *in* $0\leqslant\gamma<1$. *Here the constant* C_γ *depends only on the constants* θ_1, M, K *and* N *in* (A0), (A2) *and* (A3), *on* T *and on the parameter* γ.

In below we give an outline of the proof of Theorem 3. We omit here the proof of Theorem 2 which may not be so trivial but is rather straight-forward.

3. Outline of the proof of Theorem 3. Put

(3.1) $$u(t)=U(t,0)\varphi$$

(3.2) $$u^\tau(t)=(1+\tau A(n\tau))^{-1}(1+\tau A((n-1)\tau))^{-1}\cdots(1+\tau A(\tau))^{-1}\varphi \qquad (t=n\tau)$$

and

(3.3) $$e^\tau(t)=u^\tau(t)-u(t) \qquad (t=n\tau).$$

We can derive the following equality (3.4) whose proof is omitted:

(3.4) $$e^\tau(t_n)=-\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}(1+\tau A(t_n))^{-1}(1+\tau A(t_{n-1}))^{-1}\cdots(1+\tau A(t_k))^{-1} \times[A(t_k)U(t_k,0)-A(r)U(r,0)]\varphi dr,$$

where $t_k=k\tau$. We now examine the operator

$$A(t_k)U(t_k,0)-A(r)U(r,0)$$

and the operator

$$(1+\tau A(t_n))^{-1}(1+\tau A(t_{n-1}))^{-1}\cdots(1+\tau A(t_k))^{-1}.$$

For these operators we claim following Propositions 1 and 2, respectively.

Propositon 1. *Under the conditions* (A0)–(A3) *in Theorem* 1, *we have*

(3.5) $$\begin{aligned} &A(t+\Delta t)U(t+\Delta t,s)-A(t)U(t,s)\\ &=\frac{1}{2\pi\sqrt{-1}}\int_\Gamma \lambda e^{-(t-s)\lambda}[(\lambda-A(t+\Delta t))^{-1}-(\lambda-A(t))^{-1}]d\lambda\\ &\quad+A(t+\Delta t)[e^{-(t+\Delta t-s)A(t+\Delta t)}-e^{-(t-s)A(t+\Delta t)}]+\tilde V(t,s;\Delta t)\\ &\qquad (T\geqslant t+\Delta t>t>s\geqslant 0)\end{aligned}$$

with

(3.6) $$\|\tilde V(t,s;\Delta t)\|\leqslant C_\gamma \Delta t^\gamma\{(t-s)^{\rho-\gamma-1}+(t-s)^{\alpha-\gamma-1}\} \qquad (0<\gamma<\alpha,\rho)$$

for each γ. *Here* Γ *is the positively oriented boundary, running from* $+\infty e^{\sqrt{-1}\theta_1}$ *to* $+\infty e^{-\sqrt{-1}\theta_1}$ *of the sector* Σ:

$$\Sigma=\{\lambda\in \boldsymbol{C};|\arg\lambda|\leqslant\theta_1\}.$$

The constant C_γ *in* (3.6) *depends only on the constants* $\theta_1, M, K, \alpha, N, \rho, T$ *and* γ.

Proposition 2. *Under the conditions* (A0)–(A3) *in Theorem* 1 *with* $\rho=1$, *we have the estimate*

(3.7) $$\|(1+\tau A(t_n))^{-1}(1+\tau A(t_{n-1}))^{-1}\cdots(1+\tau A(t_{k+1}))^{-1}A(t_{k+1})^{1-\beta}\| \leqslant C_\beta \tau^{\beta-1}(n-k)^{\beta-1}$$

for each β *in* $0<\beta\leqslant 1$. *The constant* C_β *depends only on* $\theta_1, M, K, \alpha, N, T$ *and* β.

To prove Proposition 1, we just need a refined version of the method by Kato-Tanabe [4] for the construction of evolution operators $U(t,s)$. Proposition 2 can be proved by adopting Levy-Tanabe's method. Namely, in dealing with our discrete case, we imitate the method by Kato-Tanabe [4] which was originally employed in order to prove the inequality

(3.8) $$\|A(t)U(t,s)\|\leqslant C(t-s)^{-1}.$$

Now, putting

(3.9) $$e^\tau(t)=E^\tau(t)\varphi,$$

we have

(3.10) $$-E^\tau(t_n)=\sum_{k=1}^n F_1(k)+\sum_{k=1}^n F_2(k)+\sum_{k=1}^n F_3(k)$$

with

(3.11) $$F_1(k)=\int_{t_{k-1}}^{t_k} dr(1+\tau A(t_n))^{-1}\cdots(1+\tau A(t_k))^{-1} \times\frac{1}{2\pi\sqrt{-1}}\int_\Gamma \lambda e^{-r\lambda}[(\lambda-A(t_k))^{-1}-(\lambda-A(r))^{-1}]d\lambda$$

(3.12) $$F_2(k)=\int_{t_{k-1}}^{t_k} dr(1+\tau A(t_n))^{-1}\cdots(1+\tau A(t_k))^{-1}A(t_k)^{1-\beta} \times A(t_k)^\beta[e^{-t_kA(t_k)}-e^{-rA(t_k)}]$$

and

(3.13) $$F_3(k)=\int_{t_{k-1}}^{t_k} dr(1+\tau A(t_n))^{-1}\cdots(1+\tau A(t_k))^{-1}\tilde{V}(r,0\,;t_k-r),$$

because of (3.4) and (3.5).

$F_3(k)$ is estimated as follows by Proposition 1:

(3.14) $$\|F_3(k)\|\leqslant C_\gamma \tau k^{-\gamma},$$

which yields

(3.15) $$\sum_{k=1}^n \|F_3(k)\|\leqslant C_\gamma n^{-\gamma}.$$

We can estimate $F_2(k)$ as

(3.16) $$\|F_2(k)\|\leqslant C_{\beta,\gamma}(n+1-k)^{\beta-1}k^{-\beta-\gamma}$$

by taking the parameter $\beta>0$ in Proposition 2 so small that $\beta+\gamma<1$ for the given γ. Hence we have

(3.17) $$\sum_{k=1}^n \|F_2(k)\|\leqslant C_\gamma n^{-\gamma}.$$

$F_1(k)$ is estimated as follows if $k\geqslant 2$:

(3.18) $$\|F_1(k)\|\leqslant C\int_{t_{k-1}}^{t_k} r^{-1}(t_k-1)dr\leqslant C\tau k^{-1}.$$

We can derive also (3.18) for $k=1$ by a standard technique of telescoping. Hence we end up with

$$(3.19)\qquad \sum_{k=1}^{n} \|F_1(k)\| \leqslant C_\gamma n^{-\gamma}.$$

Proofs of Propositions 1 and 2 will be given in a forthcoming paper along with detailed proofs and generalization of Theorems 2 and 3 which can cover also the case of the Neumann boundary condition.

A note added. Recently the author succeeded in proving the inequality (1.14) for $\gamma=0$, which generalizes a result of Fujita-Mizutani [1] in the case of t-independence of $a_t(\, , \,)$. The proof is based on a refined study of fractional powers of operators and evolution operators. Details will be given in the paper mentioned above.

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