# 80. *Perturbation of Domains and Green Kernels of Heat Equations*

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§1. **Introduction.** Let $\Omega$ be a bounded domain in $\mathbf{R}^n$ with $C^\infty$ boundary $\gamma$. Let $\rho(x)$ be a $C^\infty$ function on $\gamma$ and $\nu_x$ be the exterior unit normal vector at $x \in \gamma$. For sufficiently small $\varepsilon \geq 0$, let $\Omega_\varepsilon$ be the bounded domain whose boundary $\gamma_\varepsilon$ is defined by

$$\gamma_\varepsilon = \{x + \varepsilon\rho(x)\nu_x \,;\, x \in \gamma\}.$$

Let $G_\varepsilon(x, y)$ be the Green function of the Dirichlet boundary value problem for the Laplacian, that is, $G_\varepsilon(x, y)$ has the following properties:

$$\begin{cases} -\Delta_x G_\varepsilon(x, y) = \delta(x-y), & x, y \in \Omega_\varepsilon \\ \quad G_\varepsilon(x, y) = 0, \quad x \in \gamma_\varepsilon, & y \in \bar{\Omega}_\varepsilon. \end{cases}$$

We abbreviate $G_0(x, y)$ as $G(x, y)$. Put

$$\delta G(x, y) = \lim_{\varepsilon \to +0} \varepsilon^{-1}(G_\varepsilon(x, y) - G(x, y)),$$

for $x, y \in \Omega$. Then, the famous Hadamard's variational formula is the following:

$$\delta G(x, y) = \int_\gamma \frac{\partial G(x, z)}{\partial \nu_z} \frac{\partial G(y, z)}{\partial \nu_z} \rho(z) d\sigma_z,$$

where $\dfrac{\partial}{\partial \nu_z}$ denotes the exterior normal derivative with respect to $z$ and $d\sigma_z$ denotes the natural surface element of $\gamma$. The readers may refer to [5], [3] and [2].

Let $0 > \lambda_1(\varepsilon)$ be the first eigenvalue of the Laplacian with Dirichlet boundary condition at $\gamma_\varepsilon$. In [4], Garabedian-Schiffer proved the following:

$$\lim_{\varepsilon \to +0} \varepsilon^{-1}(\lambda_1(\varepsilon) - \lambda_1(0)) = \int_\gamma \left(\frac{\partial \varphi_1}{\partial \nu_z}(z)\right)^2 \rho(z) d\sigma_z, \tag{1}$$

where $\varphi_1(z)$ denotes the eigen-function associated with $\lambda_1$ which satisfies

$$\int_\gamma \varphi_1(z)^2 dz = 1.$$

Their method to obtain (1) depends essentially on the simplicity of the first eigenvalue, that is,

$$\dim \{\psi \in C^\infty(\Omega_\varepsilon) \,;\, (-\Delta - \lambda_1(\varepsilon))\psi(x) = 0 \text{ in } \Omega_\varepsilon \text{ and } \psi(x) = 0 \text{ on } \gamma_\varepsilon\} = 1,$$

for any sufficiently small $\varepsilon$. The following problem occurs.

**Problem (P).** How can one treat the variational formula of eigenvalues if $-\Delta$ with Dirichlet condition at $\gamma$ has multiple eigenvalues?

In § 2, we shall give a version of Hadamard's formula for the Green kernel of heat equation in Theorem 1, and we shall give the variational formula for the trace $T_\gamma(t)$ of them in Theorem 2. The author considers Theorem 2 as a starting point of treating Problem (P). What makes our study of Problem (P) systematic is the use of the trace $T_\gamma(t)$, since it can be intrinsically defined via Green kernels of heat equation even if $-\Delta$ has multiple eigenvalues.

In § 3, we write down explicitly the variation $\delta T_\gamma(t)$ of the trace. It is well known that the trace $T_\gamma(t)$ contains information on the asymptotic behaviour of eigenvalues. Formula (3) in Theorem 3 represents $\delta T_\gamma(t)$ as a Dirichlet series whose coefficients are explicitly determined by eigen-functions. Therefore, we can get the asymptotic behaviour of eigen-functions by well known Hardy-Littlewood-Karamata's Tauberian theorem [1].

Proofs of the theorems given below are too long to be presented here, so they will be given elsewhere.

**§ 2. Hadamard's variational formula for the Green kernel of heat equations.** Let $U_\varepsilon(x, y, t)$ denote the fundamental solution of the heat equation with Dirichlet boundary condition, that is, $U_\varepsilon(x, y, t)$ has the following properties:

$$\begin{cases} (\partial_t - \Delta_x) U_\varepsilon(x, y, t) = 0, & x, y \in \Omega_\varepsilon, t > 0 \\ U_\varepsilon(x, y, t) = 0, & x \in \gamma_\varepsilon, y \in \bar{\Omega}_\varepsilon, t > 0 \\ \lim_{t \to +0} U_\varepsilon(x, y, t) = \delta(x-y), & x, y \in \Omega_\varepsilon. \end{cases}$$

We abbreviate $U_0(x, y, t)$ as $U(x, y, t)$. Put

$$\delta U(x, y, t) = \lim_{\varepsilon \to +0} \varepsilon^{-1}(U_\varepsilon(x, y, t) - U(x, y, t)).$$

Then, we have the following

**Theorem 1.** *We have*

$$\delta U(x, y, t) = \int_0^t d\tau \int_\gamma \frac{\partial U(x, z, t-\tau)}{\partial \nu_z} \frac{\partial U(y, z, \tau)}{\partial \nu_z} \rho(z) d\sigma_z.$$

Let $T_\gamma(t; \varepsilon)$ denote the trace of $U_\varepsilon(x, y, t)$ on $\Omega_\varepsilon$ which is defined by

$$T_\gamma(t; \varepsilon) = \int_{\Omega_\varepsilon} U_\varepsilon(x, x, t) dx.$$

Then, we obtain the following

**Theorem 2.** *For any fixed* $t > 0$, *we have*

$$\delta T_\gamma(t) = \int_\Omega \delta U(x, x, t) dx, \tag{2}$$

*where we put*

$$\delta T_\gamma(t) = \lim_{\varepsilon \to 0} \varepsilon^{-1}(T_\gamma(t; \varepsilon) - T_\gamma(t; 0)).$$

Note that the formula (2) assures the commutativity of $\delta$ and

$\int dx$, that is,

$$\delta \int_{\Omega} U(x, x, t)dx = \int_{\Omega} \delta U(x, x, t)dx.$$

**§ 3. Application to the asymptotics of eigen-functions.** Let $0 > \lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots$ be the eigenvalues of the Laplacian with Dirichlet boundary condition on $\gamma$. We arrange them repeatedly according to their multiplicities. Let $\{\varphi_i(x)\}_{i=1}^{\infty}$ be an orthonormal basis of $L^2(\Omega)$ consisting of eigen-functions of Laplacian. We assume that $\varphi_j(x)$ belongs to the eigen-space associated with $\lambda_j$. It is well known that

$$U(x, y, t) = \sum_{j=1}^{\infty} e^{\lambda_j t} \varphi_j(x) \varphi_j(y).$$

Therefore, by Theorems 1 and 2, we have the following

**Theorem 3.** *We have*

$$\delta T_\gamma(t) = t \sum_{j=1}^{\infty} e^{\lambda_j t} \int_{\gamma} \left( \frac{\partial \varphi_j}{\partial \nu_z}(z) \right)^2 \rho(z) d\sigma_z. \tag{3}$$

The following asymptotic expansion of $T_\gamma(t\,;\varepsilon)$ was studied by [6]. When $t$ tends to zero, we have

$$T_\gamma(t\,;\varepsilon) = a_n(\varepsilon)t^{-n/2} + a_{n-1}(\varepsilon)t^{-n/2+1/2} + \cdots + a_{n-k+1}(\varepsilon)t^{-n/2+(k-1)/2} + O(t^{-n/2+k/2}),$$

for any positive integer $k$, here $a_{n-k}(\varepsilon)$ is a function of $\varepsilon$.

Concerning $\delta T_\gamma(t)$, we can prove the following

**Theorem 4.** *When $t$ tends to zero, then we have*

$$\delta T_\gamma(t) = b_n t^{-n/2} + b_{n-1} t^{-n/2+1/2} + \cdots + b_{n-k+1} t^{-n/2+(k-1)/2} + O(t^{-n/2+k/2}).$$

The following question occurs. Can we say that the formal termwise differentiation (4) is valid?

$$b_{n-k} = \lim_{\varepsilon \to 0} \varepsilon^{-1}(a_{n-k}(\varepsilon) - a_{n-k}(0)), \qquad k = 0, 1, 2, \cdots. \tag{4}$$

In the case $k=0$, we can answer to this question affirmatively.

**Theorem 5.** *We have*

$$b_n = \lim_{\varepsilon \to 0} \varepsilon^{-1}(a_n(\varepsilon) - a_n(0)).$$

It is well known that $a_n(\varepsilon) = C_n |\Omega_\varepsilon|$, here $C_n = (2\sqrt{\pi})^{-n}$ and $|\Omega_\varepsilon|$ denotes the volume of $\Omega_\varepsilon$. We see that

$$\lim_{\varepsilon \to 0} \varepsilon^{-1}(|\Omega_\varepsilon| - |\Omega|) = \int_{\gamma} \rho(z) d\sigma_z.$$

Therefore, we obtain

**Corollary.** *We have*

$$\sum_{j=1}^{\infty} e^{\lambda_j t} \int_{\gamma} \left( \frac{\partial \varphi_j}{\partial \nu_z}(z) \right)^2 \rho(z) d\sigma_z = C_n \cdot \int_{\gamma} \rho(z) d\sigma_z \cdot t^{-1-n/2} + O(t^{-n/2-1/2}).$$

We can localize this formula by using the calculas of pseudo-differential operators such as in [7].

**Theorem 6.** *We get*

$$\sum_{j=1}^{\infty} e^{\lambda_j t}\left(\frac{\partial \varphi_j}{\partial \nu_z}(z)\right)^2\Big|_{z\in\gamma} = C_n t^{-1-n/2} + O(t^{-n/2-1/2}). \tag{5}$$

It should be noticed that the Landau's symbol in (5) can be taken to be uniform with respect to $z$.

By well known Hardy-Littlewood-Karamata's Tauberian theorem [1], we have the following

**Thoerem 7.** *For* $\lambda \mapsto +\infty$, *we have*

$$\sum_{-\lambda_j<\lambda}\left(\frac{\partial \varphi_j}{\partial \nu_z}(z)\right)^2\Big|_{z\in\gamma} = (n\,!)^{-1} C_n \lambda^{1+n/2} + o(\lambda^{n/2+1}).$$

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