79. Asymptotic Behavior of Iterates of Nonexpansive Mappings in Banach Spaces. II

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1. Introduction. Let X be a Banach space and let X^* be the dual space of X. The value of $x^* \in X^*$ at $x \in X$ will be denoted by (x, x^*). The *duality mapping* F (multi-valued) from X into X^* is defined by

$$F(x)=\{x^* \in X^* : (x, x^*)=\|x\|^2 \text{ and } \|x^*\|=\|x\|\} \qquad \text{for } x \in X.$$

We say that X is *smooth*, if $\lim_{t\to 0} t^{-1}(\|x+ty\|-\|x\|)$ exists for every x and y with $\|x\|=\|y\|=1$. F is single-valued if and only if X is smooth. The duality mapping F of a smooth Banach space X is said to be *weakly continuous* at 0 if w-$\lim_{n\to\infty} x_n=0$ in X implies that $\{F(x_n)\}$ converges weakly* to 0 in X^*, where w-$\lim_{n\to\infty} x_n$ denotes the weak limit of $\{x_n\}$. It is easy to see that Hilbert space and (l^p), $1<p<\infty$, have this property.

Throughout the rest of this paper it is assumed that X is a smooth and uniformly convex real Banach space having the duality mapping F which is weakly continuous at 0, and C is a nonempty closed convex subset of X. By $T \in \mathrm{Cont}\,(C)$ we mean that T is a nonexpansive mapping from C into itself, i.e., $T: C\to C$ satisfies $\|Tx-Ty\|\leqq\|x-y\|$ for all $x, y \in C$. The set of fixed-points of T will be denoted by $\mathcal{F}(T)$.

In [5], Z. Opial proved the following: *Let $T \in \mathrm{Cont}\,(C)$ and $x \in C$. If $\mathcal{F}(T)\neq\phi$ and $\lim_{n\to\infty}\|T^{n+1}-T^n x\|=0$, then the sequence $\{T^n x\}$ is weakly convergent to an elemet of $\mathcal{F}(T)$.* (Let F_μ be a duality mapping of X into X^* with gauge function μ (see [5]). We note here that F_μ is weakly continuous at 0 if and only if F is weakly continuous at 0.) The purpose of this note is to prove the following

Theorem. *Let $T \in \mathrm{Cont}\,(C)$ and $x \in C$. Then w-$\lim_{n\to\infty} T^n x$ exists if and only if $\mathcal{F}(T)\neq\phi$ and w-$\lim_{n\to\infty}(T^{n+1}x-T^n x)=0$. Moreover w-$\lim_{n\to\infty} T^n x \in \mathcal{F}(T)$ if the weak limit exists.*

In the case that X is a Hilbert space, the theorem has been obtained by R. E. Bruck [2].

2. Proof of Theorem. In the preceding paper [4] the author proved the following: *Let $T \in \mathrm{Cont}\,(C)$ and $x \in C$. Then w-$\lim_{n\to\infty} T^n x$ exists if and only if $\mathcal{F}(T)\neq\phi$ and $\omega_w(x)\subset\mathcal{F}(T)$, where $\omega_w(x)$ denotes the set of weak subsequential limits of $\{T^n x\}$.* Therefore to prove Theorem it suffices to show the following

Proposition. *Let* $T \in \mathrm{Cont}(C)$ *and* $x \in C$. *If*

(1) $w\text{-}\lim_{n\to\infty}(T^{n+1}x - T^n x) = 0$,

then $\omega_w(x) \subset \mathcal{F}(T)$.

Recall that X is called uniformly convex if the modulus of convexity

$$\delta(\varepsilon) = \inf\{1 - \|x+y\|/2 : \|x\| \leqq 1,\ \|y\| \leqq 1 \text{ and } \|x-y\| \geqq \varepsilon\}$$

is positive for every ε with $0 < \varepsilon \leqq 2$. Let $\alpha > 0$. It is easily seen that for every ε with $0 < \varepsilon \leqq 2\alpha$

(2) $\|x\| \leqq \alpha$, $\|y\| \leqq \alpha$ and $\|x-y\| \geqq \varepsilon$ imply $\|x+y\|/2 \leqq \alpha(1-\delta(\varepsilon/\alpha))$.

Let $\{x_n\}$ be a bounded sequence in C. Then there exists a unique point $c \in C$ such that

$$\limsup_{n\to\infty}\|x_n - c\| < \limsup_{n\to\infty}\|x_n - x\| \qquad \text{for } x \in C\backslash\{c\}.$$

(See [3].) The point c is called the *asymptotic center* of $\{x_n\}$ with respect to C. By the weak continuity of F at 0 we have the following (see [4, Lemma (b)]):

(3) Let $\{x_n\}$ be a sequence in C. If $w\text{-}\lim_{n\to\infty} x_n$ exists, then the weak limit is the asymptotic center of $\{x_n\}$ with respect to C.

Proof of Proposition. Let $u \in \omega_w(x)$. Then there is a subsequence $\{n_k\}$ of $\{n\}$ such that $w\text{-}\lim_{k\to\infty} T^{n_k}x = u$. By (1) we have

$$w\text{-}\lim_{k\to\infty} T^{n_k+m}x = u \qquad \text{for every nonnegative integer } m.$$

It follows from (3) that for every $m \geqq 0$, u is the asymptotic center of $\{T^{n_k+m}x\,;\, k=1,2,\cdots\}$ with respect to C. Consequently

(4) $\limsup_{k\to\infty}\|T^{n_k+m}x - u\| \leqq \limsup_{k\to\infty}\|T^{n_k+m}x - z\|$

for $z \in C$ and $m = 0, 1, \cdots$.

Put

$$r_m = \limsup_{k\to\infty}\|T^{n_k+m}x - u\| \qquad \text{for } m = 0, 1, 2, \cdots.$$

Then by (4) and $T \in \mathrm{Cont}(C)$ we have

$$r_{m+1} = \limsup_{k\to\infty}\|T^{n_k+m+1}x - u\| \leqq \limsup_{k\to\infty}\|T^{n_k+m+1}x - Tu\|$$
$$\leqq \limsup_{k\to\infty}\|T^{n_k+m}x - u\| = r_m \qquad \text{for } m = 0, 1, 2, \cdots.$$

Therefore $\{r_m\}$ is convergent to $r = \inf\{r_m : m \geqq 0\}$.

We now prove that u is a fixed-point of T. First, let $r = 0$. Since

$$|(u - Tu, x^*)| \leqq 2\|x^*\|\,\|T^{n_k+m}x - u\| + |(T^{n_k+m}x - T^{n_k+m+1}x, x^*)| \qquad \text{for } x^* \in X^*,$$

it follows from (1) that

$$|(u - Tu, x^*)| \leqq 2\|x^*\| \limsup_{k\to\infty}\|T^{n_k+m}x - u\| = 2\|x^*\|\, r_m$$

for every $x^* \in X^*$ and $m \geqq 0$. By $\lim_{m\to\infty} r_m = r = 0$ we have

$$(u - Tu, x^*) = 0 \qquad \text{for every } x^* \in X^*, \text{ i.e., } Tu = u.$$

Next, let $r > 0$. We use the same argument as in the proof of Theorem in [1]. To prove $u \in \mathcal{F}(T)$ it suffices to show that $\|T^p u - u\| \to 0$ as $p \to \infty$. Suppose, for contradiction, that the sequence $\{\|T^p u - u\|\}$ does not converge to 0. Then there is a $d > 0$ and a subsequence $\{p_j\}$ of $\{p\}$ such that $r \geqq d$ and $\|T^{p_j}u - u\| \geqq d$ for all $j \geqq 1$. We can choose an $\varepsilon_0 > 0$ such that $(r+\varepsilon_0)[1 - \delta(d/(r+\varepsilon_0))] < r$. By $r = \lim_{m\to\infty} r_m$ there exists a positive integer m_0 such that

$$\limsup_{k\to\infty} \|T^{n_k+m}x-u\| = r_m < r+\varepsilon_0 \qquad \text{for } m \geqq m_0.$$

Therefore for every $m \geqq m_0$ there exists a positive integer $k(m)$ such that

$$(5) \quad \|T^{n_k+m}x-u\| < r+\varepsilon_0 \qquad \text{for every } k \geqq k(m).$$

Take an integer $j>0$ with $p_j \geqq m_0$. By (5) we have that

$$\|T^{p_j}u - T^{n_k+2p_j}x\| \leqq \|u - T^{n_k+p_j}x\| < r+\varepsilon_0 \qquad \text{for } k \geqq k(p_j)$$

and

$$\|u - T^{n_k+2p_j}x\| < r+\varepsilon_0 \qquad \text{for } k \geqq k(2p_j).$$

Since $\|(T^{p_j}u - T^{n_k+2p_j}x) - (u - T^{n_k+2p_j}x)\| = \|T^{p_j}u - u\| \geqq d$, it follows from (2) that

$$\begin{aligned}\|T^{n_k+2p_j}x - (u+T^{p_j}u)/2\| &= \|(T^{p_j}u - T^{n_k+2p_j}x) + (u - T^{n_k+2p_j}x)\|/2 \\ &\leqq (r+\varepsilon_0)[1-\delta(d/(r+\varepsilon_0))] \qquad \text{for } k \geqq \max\{k(p_j), k(2p_j)\},\end{aligned}$$

and hence

$$\limsup_{k\to\infty} \|T^{n_k+2p_j}x - (u+T^{p_j}u)/2\| \leqq (r+\varepsilon_0)[1-\delta(d/(r+\varepsilon_0))] < r.$$

Since u is the asymptotic center of $\{T^{n_k+2p_j}x\,;\, k=1,2,\cdots\}$ with respect to C, we have

$$\begin{aligned}r_{2p_j} &= \limsup_{k\to\infty} \|T^{n_k+2p_j}x - u\| \\ &\leqq \limsup_{k\to\infty} \|T^{n_k+2p_j}x - (u+T^{p_j}u)/2\| < r.\end{aligned}$$

This contradicts $r = \inf\{r_m : m \geqq 0\}$. Therefore $\|T^p u - u\| \to 0$ as $p \to \infty$ and hence $u \in \mathcal{F}(T)$. Q.E.D.

3. An extension of Theorem. A mapping $T: C \to C$ is called *asymptotically nonexpansive* if there exists a sequence $\{a_n\}$ of positive numbers with $\lim_{n\to\infty} a_n = 1$ such that

$$\|T^n x - T^n y\| \leqq a_n \|x-y\| \qquad \text{for } x, y \in C \text{ and } n=1,2,\cdots.$$

S. C. Bose [1] has extended Opial's theorem (which is stated in Introduction) to the case of asymptotically nonexpansive mapping. We can also extend our Theorem to the following form:

Theorem′. *Let $T: C \to C$ be an asymptotically nonexpansive mapping and let $x \in C$. Then $w\text{-}\lim_{n\to\infty} T^n x$ exists if and only if $\mathcal{F}(T) \neq \phi$ and $w\text{-}\lim_{n\to\infty}(T^{n+1} - T^n x) = 0$. Moreover $w\text{-}\lim_{n\to\infty} T^n x \in \mathcal{F}(T)$ if the weak limit exists.*

Sketch of Proof. It suffices to prove the following (a) and (b):

(a) $w\text{-}\lim_{n\to\infty} T^n x$ exists if and only if $\mathcal{F}(T) \neq \phi$ and $\omega_w(x) \subset \mathcal{F}(T)$;

(b) if $w\text{-}\lim_{n\to\infty}(T^{n+1}x - T^n x) = 0$ then $\omega_w(x) \subset \mathcal{F}(T)$.

A proof of (a) may be found in [1]. To prove (b), let $w\text{-}\lim_{k\to\infty} T^{n_k}x = u$ and put $r_m = \limsup_{k\to\infty} \|T^{n_k+m}x - u\|$ for $m \geqq 0$. Noting that u is the asymptotic center of $\{T^{n_k+m}x\,;\, k=1,2,\cdots\}$ with respect to C for every $m \geqq 0$ and T is asymptotically nonexpansive, we have

$$r_{m+l} \leqq \limsup_{k\to\infty} \|T^{n_k+m+l}x - T^l u\| \leqq a_l r_m \qquad \text{for } m \geqq 0 \text{ and } l \geqq 0.$$

It follows from $\lim_{l\to\infty} a_l = 1$ that $\limsup_{l\to\infty} r_l = \limsup_{l\to\infty} r_{m+l} \leqq r_m$ for

$m \geqq 0$. Thus $\limsup_{l\to\infty} r_l \leqq \liminf_{m\to\infty} r_m$, and therefore $\{r_m\}$ is convergent. Put $r=\lim_{m\to\infty} r_m$. Then, using the same argument as in the proof of Proposition, we obtain that $u \in \mathcal{F}(T)$. (In this case, replace "$r_m < r+\varepsilon_0$ for $m \geqq m_0$" in the proof of Proposition by "$r_m < r+\varepsilon_0/2$ for $m \geqq m_0$". After this our argument is as follows. For every $m \geqq m_0$ there is an integer $k(m)>0$ such that $\|T^{n_k+m}x-u\|<r+\varepsilon_0/2$ for $k \geqq k(m)$. Choose an integer $j_0>0$ such that $p_{j_0} \geqq m_0$ and $a_{p_j}(r+\varepsilon_0/2)<r+\varepsilon_0$ for $j \geqq j_0$. We have that $\|T^{p_j}u-T^{n_k+2p_j}x\| \leqq a_{p_j}(r+\varepsilon_0/2)<r+\varepsilon_0$ for $k \geqq k(p_j)$ and $j \geqq j_0$, and $\|u-T^{n_k+2p_j}x\|<r+\varepsilon_0$ for $k \geqq k(2p_j)$. These and $\|T^{p_j}u-u\| \geqq d$ yield $\|T^{n_k+2p_j}x-(u+T^{p_j}u)/2\| \leqq (r+\varepsilon_0)[1-\delta(d/(r+\varepsilon_0))]$ for $k \geqq \max\{k(p_j), k(2p_j)\}$ and $j \geqq j_0$. Therefore $r_{2p_j} \leqq \limsup_{k\to\infty} \|T^{n_k+2p_j}x-(u+T^{p_j}u)/2\| < (r+\varepsilon_0)[1-\delta(d/(r+\varepsilon_0))]$ $(<r)$ for $j \geqq j_0$. This contradicts $r=\lim_{m\to\infty} r_m$.)

References

[1] S. C. Bose: Weak convergence to the fixed point of an asymptotically nonexpansive map. Proc. Amer. Math. Soc., **68**, 305–308 (1978).

[2] R. E. Bruck: On the almost-convergence of iterates of a nonexpansive mappings in Hilbert space and the structure of the weak ω-limit set. Israel J. Math., **29**, 1–16 (1978).

[3] M. Edelstein: Fixed point theorems in uniformly convex Banach spaces. Proc. Amer. Math. Soc., **44**, 369–374 (1974).

[4] I. Miyadera: Asymptotic behavior of iterates of nonexpansive mappings in Banach spaces. Proc. Japan Acad., **54**, 212–214 (1978).

[5] Z. Opial: Weak convergence of the sequence of successive approximations for nonexpansive mappings. Bull. Amer. Math. Soc., **73**, 591–597 (1967).