74. An Extendability Criterion for Vector Bundles on Ample Divisors

By Takao FUJITA
Department of Mathematics, College of General Education,
University of Tokyo

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In this note we remark that the method of Grothendieck for the extension of line bundles (see SGA2 [1] or Hartshorne [2, Chap. IV]) can be applied also for vector bundles after a slight modification. Details and proofs shall be published elsewhere.

Proposition A. *Let A be a non-singular ample divisor on a manifold M. Let E be a vector bundle on A. Suppose that $H^2(A, \mathcal{E}nd(E)\otimes[tA]_A)=0$ for any $t<0$. Then E can be extended to a vector bundle on the formal completion $\hat{M}$ of M along A.*

Proposition B. *Let A, M and $\hat{M}$ be as above. Let $\hat{E}$ be a vector bundle on $\hat{M}$ and put $E=\hat{E}_A$. Suppose that $\dim A\geqq 2$ and $H^p(A, E\otimes[tA]_A)=0$ for any integer t, p with $0<p<\dim A$. Then $\hat{E}$ can be extended to a vector bundle on M.*

Main theorem. *Let A be a non-singular ample divisor on a manifold M with $\dim M\geqq 3$. Let E be a vector bundle on A such that $H^2(A, \mathcal{E}nd(E)\otimes[-tA]_A)=0$ for any $t>0$ and that $H^p(A, E\otimes[tA]_A)=0$ for any integer t, p with $0<p<\dim A$. Then E can be extended to a vector buudle $\tilde{E}$ on M.*

Remark. In the above situation, one can prove that $H^2(M, \mathcal{E}nd(\tilde{E})\otimes[-tA])=0$ for any $t>0$ and that $H^p(M, \tilde{E}\otimes[tA])=0$ for any integer t, p with $0<p<\dim M$.

Combining the result of Sato [4], we obtain the following

Theorem. *Let E be a vector bundle on a manifold M with $\dim M \geqq 3$ which is a complete intersection in a projective space $\boldsymbol{P}^N$. Then E is a direct sum of line bundles if and only if the following two conditions are satisfied.*

a) $H^2(M, \mathcal{E}nd(E)(-t))=0$ *for any* $t>0$.

b) $H^p(M, E(t))=0$ *for any* t, p *with* $0<p<\dim M$.

Remark. The above condition a) is indispensable. Indeed, let M be the grassmannian variety of the lines on $\boldsymbol{P}^3$. Then the Plücker embedding makes M a smooth hyperquadric in $\boldsymbol{P}^5$. The tautological vector bundle E on M satisfies the condition b), but it is not decomposable.

Remark. Let A be a hyperplane in $M=\boldsymbol{P}^{n+1}$. Then any vector bundle E on A can be extended to $\hat{M}$ by a projection map $M-\{x\}\to A$ from a point $x \in M-A$. Hence the condition a) is not necessary in order to apply Proposition B. Thus we obtain a new proof of the following

Theorem (Horrocks [3]). *Let E be a vector bundle on $\boldsymbol{P}^n$. Then E is a direct sum of line bundles if and only if $H^p(\boldsymbol{P}^n, E(t))=0$ for any t, p with $0<p<n$.*

This work of Horrocks suggests further development of the study of the obstruction to extending vector bundles.

References

[1] A. Grothendieck: Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux. SGA 2, North Holland Publ. Comp., Amsterdam (1968).

[2] R. Hartshorne: Ample subvarieties of algebraic varieties. Lecture Notes in Math. Springer, no. 156 (1970).

[3] G. Horrocks: Vector bundles on the punctured spectrum of a local ring. Proc. London Math. Soc., **14** (3), 689–713 (1964).

[4] E. Sato: The decomposability of an infinitely extendable vector bundle on the projective space. J. Math. Kyoto Univ., **17**, 127–150 (1977).