73. A Residue Formula for Chern Classes Associated with Logarithmic Connections

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1. Let E be a holomorphic vector bundle over a complex manifold M, and D be a meromorphic connection of E. Under some assumptions below, we have a relation between the residues of D and the Chern classes of E. The purpose of this note is to state the relation. Full details and proofs will be published elsewhere.

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2. Let Z be the pole of D. We assume the following:

(H.1) *Z is normal crossing,*

(H.2) *Let $Z=\bigcup_{j\in N} Z_j$ be the decomposition of Z into irreducible components (N is a set of indices). Each Z_j is non-singular,*

(H.3) *D has a simple logarithmic pole along Z (see Deligne* [3]),

(H.4) *In the case when M is not compact, D is assumed to be integrable.*

Then the residue of D along Z_j, $\mathrm{Res}_j D$, is well-defined as a holomorphic section of $\mathrm{End}(E)|_{Z_j}$, the restriction of the endomorphism bundle of E onto Z_j [3].

Let $J=(j_1, j_2, \cdots, j_k)$ be an element of $N^k=N\times N\times\cdots\times N$ (k times). If among $j_1, j_2, \cdots, j_k$ there exists p different indices, say $j_1^*, j_2^*, \cdots, j_p^*$, put $J^*=\{j_1^*, j_2^*, \cdots, j_p^*\}$ and let a_m be the number of j_m^* appearing in J $(1\leqq m\leqq p)$.

Define $Z_{J^*}=\bigcap_{m=1}^{p} Z_{j_m^*}$. This is a submanifold of M of codimension p, not necessarily connected. Let $Z_{J^*}=\bigcup_i Z_{J^*}^{(i)}$ be the decomposition of Z_{J^*} into connected components.

It is known ([2], [3]).

Let $c_k(A_1, A_2, \cdots, A_k)$ be the completely polarized form of the k-th Chern polynomial $c_k(A)$ (A, A_j are matrices). Then for any element $J=(j_1, j_2, \cdots, j_k)$ in N^k,

$$c_k(\mathrm{Res}_{j_1} D, \mathrm{Res}_{j_2} D, \cdots, \mathrm{Res}_{j_k} D)$$

is constant on each component $Z_{J^*}^{(i)}$ by (H.4) or by the compactness of M when M is compact. Denote this value by $c_k(\mathrm{Res}_J D)^{(i)}$.

A submanifold W of M of codimension p determines an element of $H^{p,p}(M)$, the Dolbeault cohomology group of M of type (p, p). We denote it by $c_p(W)$.

3. Now we can state the following

Theorem. *In* $H^{k,k}(M)$,

$$c_k(E)=(-1)^k \sum_{J \in N^k} \left\{\sum_i \{c_k(\mathrm{Res}_J D)^{(i)} c_p(Z_{J^*}^{(i)})\} \prod_{m=1}^{p} c_1(Z_{j^*})^{a_m-1}\right\}$$

where $c_k(E)$ *is the k-th Chern class of* E.

Remarks. (1) Let K be a $\bar{\partial}$-closed smooth curvature matrix of E of type $(1,1)$. $c_k(E)$ is the class in $H^{k,k}(M)$ represented by the (k, k)-form $c_k\left(\frac{-1}{2\pi i}K\right)$.

(2) In general, $c_k(\mathrm{Res}_J D)$ takes different values on various components $Z_{J^*}^{(i)}$.

4. Let $\{U\}$ be an open covering of M such that

(1) there exists a holomorphic frame e of E on U,

(2) each Z_j is defined by a holomorphic function f_j in U.

Then it holds that $D(e)=D\otimes e$ on U, where the latter D is a matrix of 1-forms on U. The connection matrix D is explicity written for each $J \in N^k$ as follows;

$$D=\sum_{j \in J^*} A_j \frac{df_j}{f_j}+B_{J^*} \qquad \text{on } U,$$

where A_j are matrices of holomorphic functions on U and B_{J^*} is a matrix of 1-forms on U having a simple logarithmic pole along Z_s, $s \notin J^*$.

Let $[Z_j]$ be the line bundle determined by the divisor Z_j and D_j be a smooth metric connection of $[Z_j]$. Put

$$\tilde{B}_{J^*}=B_{J^*}|_{Z_{J^*}}-\sum_{j \in J^*} \mathrm{Res}_j D \cdot D_j|_{Z_{J^*}}.$$

Then $\tilde{B}_{J^*}$ is a connection of $E|_{Z_{J^*}}$ with poles along $Z_s \cap Z_{J^*}$, $s \notin J^*$.

Let D_E be a smooth metric connection of E. Put

$$L_{J^*}=\tilde{B}_{J^*}-D_E|_{Z_{J^*}} \qquad \text{on } Z_{J^*}.$$

Then we have the following

Proposition. L_{J^*} *is an* End (E)-*valued* 1-*form on* Z_{J^*} *satisfying the following*:

(1) $\bar{\partial} L_{J^*}=-\sum_{j \in J^*} \mathrm{Res}_j D \cdot K_j - K_E \qquad$ *on* $Z_{J^*}-\bigcup_{s \notin J^*} Z_s$,

(2) $\mathrm{Res}_s L_{J^*}=\mathrm{Res}_s D \qquad$ *for* $s \notin J^*$,

where K_E *(resp.* K_j*) is the curvature matrix of the connection* D_E *(resp.* D_j*).*

Using the above proposition and a similar technique used by Bott in [1], we can identify the both sides of the formula in the theorem as cohomologies defined by currents on M of type (k, k).

[1] R. Bott: Vector fields and characteristic numbers. Michigan Math. J., **14**, 231–244 (1967).
[2] S. S. Chern: Complex Manifolds without Potential Theory. Van Nostrand (1967).
[3] P. Deligne: Équations différentielles à points singuliers réguliers. Lecture Notes in Math., 163, Springer (1970).