68. *Vanishing Theorems of Cohomology Groups with Coefficients in Sheaves of Holomorphic Functions with Bounds*

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In this paper we announce vanishing theorems of cohomology groups with coefficients in sheaves of holomorphic functions with bounds. Combining those theorems with the approximation theorem of Nagamachi-Mugibayashi [6], we can deduce the fundamental properties of the sheaf of modified Fourier hyperfunctions. This will be published elsewhere.

1. Notations and definitions. We denote by $\boldsymbol{Q}^n$ the radial compactification of $\boldsymbol{C}^n \cong \boldsymbol{R}^{2n}$, and by $\boldsymbol{D}^n$ the closure of $\boldsymbol{R}^n$ in $\boldsymbol{Q}^n$. (See Nagamachi-Mugibayashi [6].)

Definition 1. We denote by $\mathcal{O}_{inc}$ the sheaf on $\boldsymbol{Q}^n$ whose section module $\mathcal{O}_{inc}(W)$ over an open set W in $\boldsymbol{Q}^n$ is given by the following:

$$\mathcal{O}_{inc}(W)=\Big\{f \in \mathcal{O}(W \cap \boldsymbol{C}^n)\,;\ \sup_{z \in K \cap \boldsymbol{C}^n} |f(z) \exp(-\varepsilon|z|)| < \infty \text{ for all } K \subset\subset W \text{ and all } \varepsilon > 0\Big\}.$$

Definition 2. We denote by $\mathcal{O}_{dec}$ the sheaf on $\boldsymbol{Q}^n$ whose section module $\mathcal{O}_{dec}(W)$ over an open set W in $\boldsymbol{Q}^n$ is given by the following:

$$\mathcal{O}_{dec}(W)=\Big\{f \in \mathcal{O}(W \cap \boldsymbol{C}^n)\,;\ \text{for all } K \subset\subset W, \text{ there exists an } \varepsilon > 0 \text{ such that } \sup_{z \in K \cap \boldsymbol{C}^n} |f(z) \exp(\varepsilon|z|)| < \infty\Big\}$$

Definition 3. We denote by $\mathcal{X}^0$ the sheaf on $\boldsymbol{Q}^n$ whose section module $\mathcal{X}^0(W)$ over an open set W in $\boldsymbol{Q}^n$ is given by the following:

$$\mathcal{X}^0(W)=\Big\{f \in L^2_{loc}(W \cap \boldsymbol{C}^n)\,;\ \int_{K \cap \boldsymbol{C}^n} |f(z)|^2 \exp(-\varepsilon|z|)\, d\lambda < \infty \text{ for all } K \subset\subset W \text{ and all } \varepsilon > 0\Big\},$$

where $d\lambda$ is the Lebesgue measure on $\boldsymbol{C}^n \cong \boldsymbol{R}^{2n}$.

Definition 4. We denote by $\mathcal{Y}^0$ the sheaf on $\boldsymbol{Q}^n$ whose section module $\mathcal{Y}^0(W)$ over an open set W in $\boldsymbol{Q}^n$ is given by the following:

$$\mathcal{Y}^0(W)=\Big\{f \in L^2_{loc}(W \cap \boldsymbol{C}^n)\,;\ \text{for all } K \subset\subset W, \text{ there exists an } \varepsilon > 0 \text{ such that } \int_{K \cap \boldsymbol{C}^n} |f(z)|^2 \exp(\varepsilon|z|)\, d\lambda < \infty\Big\}.$$

Let $\mathcal{F}$ be a sheaf of certain functions on $\boldsymbol{Q}^n$, then we denote by $\mathcal{F}_{(p,q)}$ the sheaf of differential forms of type (p, q) whose coefficients are sections of $\mathcal{F}$.

Definition 5. We denote by $\mathcal{X}^1_{(p,q)}$ the sheaf on $\boldsymbol{Q}^n$ whose section module $\mathcal{X}^1_{(p,q)}(W)$ over an open set W in $\boldsymbol{Q}^n$ is given by the following:

$$\mathcal{X}^1_{(p,q)}(W)=\{f \in \mathcal{X}^0_{(p,q)}(W)\,;\, \bar{\partial} f \in \mathcal{X}^0_{(p,q+1)}(W)\},$$

where $\bar{\partial} f$ is defined in the sense of distributions on $\boldsymbol{C}^n$.

Definition 6. We denote by $\mathcal{Y}^1_{(p,q)}$ the sheaf on $\boldsymbol{Q}^n$ whose section module $\mathcal{Y}^1_{(p,q)}(W)$ over an open set W in $\boldsymbol{Q}^n$ is given by the following:

$$\mathcal{Y}^1_{(p,q)}(W)=\{f \in \mathcal{Y}^0_{(p,q)}(W)\,;\, \bar{\partial} f \in \mathcal{Y}^0_{(p,q+1)}(W)\},$$

where $\bar{\partial} f$ is defined in the sense of distributions on $\boldsymbol{C}^n$.

Remark 1. $\mathcal{X}^1_{(p,q)}$ and $\mathcal{Y}^1_{(p,q)}$ are soft sheaves.

Remark 2. The restrictions of the sheaves $\mathcal{O}_{inc}$ and $\mathcal{O}_{dec}$ to $\boldsymbol{C}^n$ coincide with the sheaf $\mathcal{O}$ of holomorphic functions on $\boldsymbol{C}^n$. The restrictions of the sheaves $\mathcal{X}^0$ and $\mathcal{Y}^0$ to $\boldsymbol{C}^n$ coincide with the sheaf $\mathcal{L}^2_{loc}$ of locally square summable functions on $\boldsymbol{C}^n$.

Definition 7. We call an open set W in $\boldsymbol{Q}^n$ to be of *type*-1, if it satisfies the following condition:

$$\sup_{z \in W \cap \boldsymbol{C}^n} |\mathrm{Im}\, z|/(|\mathrm{Re}\, z|+a)<1 \qquad \text{for some } a>0.$$

Definition 8. We call an open set V in $\boldsymbol{Q}^n$ to be $\mathcal{O}_{inc}$-*pseudoconvex*, if it is of type-1 and there exists a strictly plurisubharmonic C^2-function $p(z)$ on $V \cap \boldsymbol{C}^n$ satisfying the following conditions:

i) $\{z \in V \cap \boldsymbol{C}^n\,;\, p(z)<c\} \subset\subset V \qquad$ for all $c \in \boldsymbol{R}$,

ii) $\sup_{z \in K \cap \boldsymbol{C}^n} p(z)<\infty \qquad$ for all $K \subset\subset V$.

Remark. Considering the function $p(z)+|z|^2$, we find that $V \cap \boldsymbol{C}^n$ is pseudoconvex, if V is an $\mathcal{O}_{inc}$-pseudoconvex open set in $\boldsymbol{Q}^n$.

2. Main theorems. Theorem 1. *For all $\mathcal{O}_{inc}$-pseudoconvex open sets V in $\boldsymbol{Q}^n$, we have* $\mathrm{H}^s(V\,;\mathcal{O}_{inc})=0$ $(s \geqslant 1)$.

Theorem 2. *For all open sets W of type-1 in $\boldsymbol{Q}^n$, we have* $\mathrm{H}^n(W\,;\mathcal{O}_{inc})=0$.

Theorem 3. *For all open sets Ω in $\boldsymbol{D}^n$, there exists a fundamental system of neighbourhoods of Ω in $\boldsymbol{Q}^n$ consisting of $\mathcal{O}_{inc}$-pseudoconvex open sets.*

3. Sketch of the proofs. In this section we will describe the outlines of the proofs of the main theorems. The details will be published elsewhere.

In order to show Theorems 1 and 2, we show the following

Theorem 4. *For all $\mathcal{O}_{inc}$-pseudoconvex open sets V in $\boldsymbol{Q}^n$, we have the following exact sequence*:

$$\mathcal{X}^0_{(p,0)}(V) \xrightarrow{\bar{\partial}} \mathcal{X}^0_{(p,1)}(V) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{X}^0_{(p,n)}(V) \longrightarrow 0.$$

Sketch of the proof. We note that $\mathcal{X}^0_{(p,q)}(V)$ becomes an FS^* space (for the definition of FS^* spaces, see p. 380 of Komatsu [5]), equipped with the following topology of a projective limit of Hilbert spaces:

$$\mathcal{X}^0_{(p,q)}(V)=\lim_j \operatorname{proj} X_{j,(p,q)},$$

where

$$X_{j,(p,q)}=L^2_{(p,q)}(\mathring{K}_j \cap \boldsymbol{C}^n\,;\,(1/j)\|z\|),$$

$\{K_j\}$ is an increasing sequence of compact subsets of V which exhausts V and the symbol $\|z\|$ is a slight modification of $|z|$ near $0 \in \boldsymbol{C}^n$ so as to become a convex C^∞-function on $\boldsymbol{C}^n$. Here we followed the notation of Hörmander [1].

Therefore by the theory of FS^* spaces of Komatsu [5], the strong dual space of $\mathcal{X}^0_{(p,q)}(V)$ is given by the following DFS^* space:

$$\mathcal{Y}^0_{comp(p,q)}(V)=\lim_j \operatorname{ind} Y_{j,(p,q)},$$

where

$$Y_{j,(p,q)}=L^2_{(p,q)}(K_j \cap \boldsymbol{C}^n\,;\,-(1/j)\|z\|).$$

Here the injection $\rho'_j : Y_{j,(p,q)} \hookrightarrow \mathcal{Y}^0_{comp(p,q)}(V)$ is given by the following:

$$\rho'_j f(x)=\begin{cases} f(x) & \text{if } x \in \mathring{K}_j \cap \boldsymbol{C}^n \\ 0 & \text{otherwise} \end{cases} \quad \text{for } f \in Y_{j,(p,q)}.$$

Next we note that, using a strictly plurisubharmonic C^2-function on $V \cap \boldsymbol{C}^n$ satisfying the conditions i) and ii) of Definition 8, we can choose an exhaustion $\{K_j\}$ so that each $K_j \cap \boldsymbol{C}^n$ has a strictly pseudoconvex C^2-boundary.

Then we can prove the theorem, combining the theory of L^2-estimates for the $\bar{\partial}$ operator of Hörmander [1], [2] with the theory of FS^* spaces and DFS^* spaces of Komatsu [5].

Proposition 5. *We have the following soft resolution of the sheaf $\mathcal{O}_{inc}$ on $\boldsymbol{Q}^n$:*

$$0 \longrightarrow \mathcal{O}_{inc} \longrightarrow \mathcal{X}^1_{(0,0)} \xrightarrow{\bar{\partial}} \mathcal{X}^1_{(0,1)} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{X}^1_{(0,n)} \longrightarrow 0. \tag{2.1}$$

Proof. First we note that for all $z \in \boldsymbol{Q}^n$, there exists a fundamental system $\{V_j\}$ of neighborhoods of z so that each UV_j is an $\mathcal{O}_{inc}$-pseudoconvex open set for a unitary matrix U. On the other hand, by Theorem 4, if $q \geqslant 1$ we have

$$\begin{aligned}\operatorname{Ker}\{&\bar{\partial} : \mathcal{X}^1_{(0,q)}(V) \to \mathcal{X}^1_{(0,q+1)}(V)\} \\ &= \operatorname{Ker}\{\bar{\partial} : \mathcal{X}^0_{(0,q)}(V) \to \mathcal{X}^0_{(0,q+1)}(V)\} \\ &= \operatorname{Im}\{\bar{\partial} : \mathcal{X}^0_{(0,q-1)}(V) \to \mathcal{X}^0_{(0,q)}(V)\} \\ &= \operatorname{Im}\{\bar{\partial} : \mathcal{X}^1_{(0,q-1)}(V) \to \mathcal{X}^1_{(0,q)}(V)\}\end{aligned}$$

for all $\mathcal{O}_{inc}$-pseudoconvex open sets V in $\boldsymbol{Q}^n$. Estimating the sup-norms by the L^2-norms, we have

$$\operatorname{Ker}\{\bar{\partial} : \mathcal{X}^1_{(0,0)}(V) \to \mathcal{X}^1_{(0,1)}(V)\} = \mathcal{O}_{inc}(V)$$

for all open sets V in $\boldsymbol{Q}^n$. So we obtain the proposition.

Proof of Theorem 1. Using the soft resolution (2.1) we have

$$\text{(2.2)} \qquad \mathrm{H}^s(V\,;\mathcal{O}_{inc})=\frac{\mathrm{Ker}\,\{\bar\partial:\mathcal{X}^1_{(0,s)}(V)\to\mathcal{X}^1_{(0,s+1)}(V)\}}{\mathrm{Im}\,\{\bar\partial:\mathcal{X}^1_{(0,s-1)}(V)\to\mathcal{X}^1_{(0,s)}(V)\}}=\frac{\mathrm{Ker}\,\{\bar\partial:\mathcal{X}^0_{(0,s)}(V)\to\mathcal{X}^0_{(0,s+1)}(V)\}}{\mathrm{Im}\,\{\bar\partial:\mathcal{X}^0_{(0,s-1)}(V)\to\mathcal{X}^0_{(0,s)}(V)\}}$$

for all open sets V in $\boldsymbol{Q}^n$. If V is $\mathcal{O}_{inc}$-pseudoconvex, we find that the left hand side of (2.2) vanishes thanks to Theorem 4. So we obtain the theorem.

Sketch of the proof of Theorem 2. It is sufficient to show

$$\mathcal{X}^1_{(0,n-1)}(W)\xrightarrow{\bar\partial^{n-1}}\mathcal{X}^1_{(0,n)}(W)\longrightarrow 0 \qquad \text{(exact)}.$$

In particular it is sufficient to show

$$\mathcal{X}^0_{(0,n-1)}(W)\xrightarrow{\bar\partial^{n-1}}\mathcal{X}^0_{(0,n)}(W)\longrightarrow 0 \qquad \text{(exact)}.$$

This can be proved using the ellipticity of the dual operator of $\bar\partial^{n-1}$, the theory of L^2-estimates for the $\bar\partial$ operator of Hörmander [1], [2] and the theory of FS^* spaces and DFS^* spaces of Komatsu [5].

Sketch of the proof of Theorem 3. We can prove the theorem in a similar way to the proof of Theorem 2.1.6 of Kawai [4] with some delicate estimates.

4. Other theorems. **Theorem 6.** *Let Ω be an open set in $\boldsymbol{D}^n$. Then we have* $\mathrm{H}^s(\Omega\,;\mathcal{O}_{inc}|_{\boldsymbol{D}^n})=0$ $(s\geqslant 1)$.

Proof. This is an immediate consequence of Theorems 1 and 3.

Next we consider to construct a soft resolution of the sheaf $\mathcal{O}_{dec}$.

Theorem 7. *Let K be a compact subset in $\boldsymbol{Q}^n$. Suppose that there exists a fundamental system of neighborhoods of K consisting of $\mathcal{O}_{inc}$-pseudoconvex open sets. Then we have the following exact sequence*:

$$\mathcal{Y}^0_{(p,0)}(K)\xrightarrow{\bar\partial}\mathcal{Y}^0_{(p,1)}(K)\xrightarrow{\bar\partial}\cdots\xrightarrow{\bar\partial}\mathcal{Y}^0_{(p,n)}(K)\longrightarrow 0.$$

Proof. Let f be a section of $\mathcal{Y}^0_{(p,q+1)}$ over K such that $\bar\partial f=0$. Now we put $g_\varepsilon(z)=\cosh(\varepsilon(z_1^2+\cdots+z_n^2)^{1/2})$. Then there exists an $\varepsilon>0$ such that $g_\varepsilon f\in\mathcal{X}^0_{(p,q+1)}(K)$. Note that $\bar\partial(g_\varepsilon f)=0$. Since K has a fundamental system of neighborhoods consisting of $\mathcal{O}_{inc}$-pseudoconvex open sets, there exists a $u\in\mathcal{X}^0_{(p,q)}(K)$ such that $\bar\partial u=g_\varepsilon f$. Then we have $\bar\partial(g_\varepsilon^{-1}u)=f$ and $g_\varepsilon^{-1}u\in\mathcal{Y}^0_{(p,q)}(K)$, which implies the theorem.

Proposition 8. *We have the following soft resolution of the sheaf* $\mathcal{O}_{dec}$:

$$0\longrightarrow\mathcal{O}_{dec}\longrightarrow\mathcal{Y}^0_{(0,0)}\xrightarrow{\bar\partial}\mathcal{Y}^0_{(0,1)}\xrightarrow{\bar\partial}\cdots\xrightarrow{\bar\partial}\mathcal{Y}^0_{(0,n)}\longrightarrow 0.$$

Estimating the sup-norms by the L^2-norms, we have

$$\mathcal{K}er\,\{\bar\partial:\mathcal{Y}^1_{(0,0)}\to\mathcal{Y}^1_{(0,1)}\}=\mathcal{O}_{dec}.$$

We can show exactness at the other terms in a similar way to the proof of Proposition 5 using Theorem 7.

We obtain also the following

Theorem 9 (Nagamachi-Mugibayashi [6, Lemma 4.9]). *Let a compact set K be as in Theorem 7. Then we have* $\mathrm{H}^s(K; \mathcal{O}_{dec})=0$ $(s \geqslant 1)$.

Remark. In a similar way to the proofs of this paper, we can improve the details of proofs of the corresponding theorems of Kawai [4] and Ito-Nagamachi [3].

References

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