59. *Absolute Continuity of Probability Laws of Wiener Functionals*

By Ichiro SHIGEKAWA
Department of Mathematics, Kyoto University
(Communicated by Kôsaku YOSIDA, M. J. A., Oct. 12, 1978)

1. The *Wiener space*, which is a typical example of abstract Wiener spaces introduced by L. Gross [1], is a triple (B, H, μ) where B is a Banach space consisting of all real valued continuous functions $x(t)$ $(x(0)=0)$ defined on the interval $[0,1]$ with norm $\|x\|=\sup_{0\leqslant t\leqslant 1}|x(t)|$, H is a Hilbert space consisting of absolutely continuous functions $x(t)$ $(x(0)=0)$ such that $x'(t)\in L^2[0,1]$ with inner product

$$\langle x, y\rangle_H=\int_0^1 x'(t)y'(t)dt$$

and μ is the Wiener measure, i.e., the Borel probability measure on B such that

$$\int_B e^{i(h,x)}\mu(dx)=\exp\left\{-\frac{1}{2}\langle h, h\rangle_H\right\}, \tag{1}$$

where $h\in B^*\subset H$ and $(\ ,\ )$ is a natural paring of B^* and B. It is readily seen that $\{x(t)\,;0\leqslant t\leqslant 1\}$ is a standard Wiener process on the probability space (B,μ). A real-valued (or more generally, a Banach space-valued) measurable function defined on the probability space (B,μ) is called a *Wiener functional.* Two Wiener functionals $F_1(x)$ and $F_2(x)$ are identified if $\mu\{x\,; F_1(x)\neq F_2(x)\}=0$. Typical examples of Wiener functionals are solutions of stochastic differential equations or multiple Wiener integrals (see Itô [2]).

Malliavin [3] introduced a notion of derivatives of Wiener functionals and applied it to the absolute continuity of the probability law induced by a solution of stochastic differential equations at a fixed time. Here, we define the derivatives of Wiener functionals in a somewhat different way and rephrase a theorem of Malliavin. We will apply it to the absolute continuity of the probability law induced by a system of multiple Wiener integrals.

2. Let (B,H,μ) be the Wiener space or more generally, any abstract Wiener space. Let E be a Banach space, F be a mapping from B into E, and $\mathcal{L}(B,E)$ denote the space of all bounded linear operators from B into E. If there exists an operator $T\in\mathcal{L}(B,E)$ such that

$$F(x+y)-F(x)=T(y)+o(\|y\|) \qquad \text{as } \|y\|\to 0\ (y\in B), \tag{2}$$

then F is said to be *B-differentiable at* $x\in B$, and the operator T is called the *B-derivative* (or Fréchet derivative) *of* F *at* $x\in B$, $F'(x)$ in notation. If F is B-differentiable at every point of B, we say simply that *F is B-differentiable.* Similarly, if there exists an operator $S \in \mathcal{L}(H, E)$ such that

(3) $\quad F(x+h)-F(x)=S(h)+o(|h|_H) \qquad$ as $|h|_H \to 0$ $(h \in H)$,

then F is said to be *H-differentiable at $x \in B$*, and the operator S is called the *H-derivative of F at $x \in B$, $DF(x)$* in notation. If F is H-differentiable at every point of B, we say that *F is H-differentiable.* Clearly if F is B-differentiable, then F is also H-differentiable, and $DF(x)=F'(x)|_H$. Inductively we can define $F'', F''', \cdots$, and $D^2F, D^3F, \cdots$. We may regard $F^{(n)}$ as an element of $\mathcal{L}^n(B, E)$ and D^nF as an element of $\mathcal{L}^n(H, E)$, where $\mathcal{L}^n(B, E)$ is a space of continuous n-linear operators from B into E, and $\mathcal{L}^n(H, E)$ is defined similarly. When E is a Hilbert space, $S \in \mathcal{L}^n(H, E)$ is said to be of *Hilbert-Schmidt class* if

(4) $$\sum_{i_1, i_2, \cdots, i_n=1}^{\infty} |S(h_{i_1}, h_{i_2}, \cdots, h_{i_n})|_E^2 < \infty$$

for any orthonormal system $\{h_i\}_{i=1}^{\infty}$ of H. We denote by $\mathcal{L}_{(2)}^n(H, E)$ the space of all $S \in \mathcal{L}^n(H, E)$ which are of Hilbert-Schmidt class. Then $\mathcal{L}_{(2)}^n(H, E)$ is a Hilbert space with its inner product given by

(5) $$\langle T, S\rangle_{\mathcal{L}_{(2)}^n(H,E)} = \sum_{i_1, i_2, \cdots, i_n=1}^{\infty} \langle T(h_{i_1}, h_{i_2}, \cdots, h_{i_n}), S(h_{i_1}, h_{i_2}, \cdots, h_{i_n})\rangle_E$$

for $T, S \in \mathcal{L}_{(2)}^n(H, E)$, where $\{h_i\}_{i=1}^{\infty}$ is a complete orthonormal system in H.

Definition 1. Let K be a Hilbert space, and F be a Wiener functional from B into K. Then $F \in H(p_0, p_1, \cdots, p_n)(K)$, $(p_0, p_1, \cdots, p_n \geqslant 1)$ if and only if F satisfies the following.

(i) $F \in L^{p_0}(\mu; K)$ and there exists a sequence $\{f_k\}_{k=1}^{\infty}$ of n times B-differentiable mappings from B into K, such that $f_k \in L^{p_0}(\mu; K)$ and $\lim_{k\to\infty} f_k = F$ in $L^{p_0}(\mu; K)$, where $L^{p_0}(\mu; K)$ is a set of all Wiener functionals $f: B \to K$ such that

(6) $$\|f\|_{L^{p_0}(\mu;K)} = \left\{\int_B |f(x)|_K^{p_0}\, \mu(dx)\right\}^{1/p_0} < \infty;$$

(ii) for $m=1, 2, \cdots, n$, $D^m f_k(x)$ belongs to $\mathcal{L}_{(2)}^m(H, K)$ for all $x \in B$ and a sequence $\{D^m f_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $L^{p_m}(\mu; \mathcal{L}_{(2)}^m(H, K))$;

(iii) for any $k=1, 2, \cdots$, there exists a finite dimensional projection Q_k such that $f_k(x)=f_k(Q_k x)$.

Then, we define D^mF as the limit of $\{D^m f_k\}_{k=1}^{\infty}$ in $L^{p_m}(\mu; \mathcal{L}_{(2)}^m(H, K))$, and call D^mF the m-th *weak H-derivative*.

The sequence $\{f_k\}$ in (i) is called an *approximating sequence.* We can easily show that D^mF does not depend on the choice of an approximating sequence and hence is well defined.

Definition 2. Let $F: B \to R$ be a twice B-differentiable function. Then the *Ornstein-Uhlenbeck operator* L is defined by

(7) $\quad (LF)(x) = \text{trace}\,(D^2F(x)) - (F'(x), x).$

Definition 3. Let F be an R-valued Wiener functional defined on B. Then $F \in H(p_0, p_1, p_2; p_L)$, $(p_0, p_1, p_2, p_L \geqslant 1)$ if and only if F satisfies the following.

(i) $F \in H(p_0, p_1, p_2)(R)$;

(ii) there exists an approximating sequence $\{f_k\}_{k=1}^{\infty}$ in $H(p_0, p_1, p_2)$ for F satisfying also that $\{Lf_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $L^{p_L}(\mu)$.

We call the limit of $\{Lf_k\}_{k=1}^{\infty}$ the *L-derivative of F*, LF in notation. (Note that the limit is independent of choice of $\{f_k\}_{k=1}^{\infty}$.)

Theorem 1. *Let $F=(F^1, F^2, \cdots, F^n)$ be an R^n-valued Wiener functional defined on B. We assume F satisfies the following.*

(i) $F^i \in H(1,2,1;1)$, $i=1,2,\cdots,n$;

(ii) $\sigma^{ij}(x)=\langle DF^i(x), DF^j(x)\rangle_H \in H(1,2,1;1)$, $i,j=1,2,\cdots n$;

(iii) $\det(\sigma^{ij}(x)) \neq 0$ μ-a.e.

Then the probability law of F is absolutely continuous with respect to Lebesgue measure on R^n.

3. We denote by $I_p(f)$ a multiple Wiener integral for $f \in \widehat{L^2}([0,1]^p)$, where $\wedge$ means the space of symmetric functions.

Theorem 2. *For every $n \in N$ and $p_0, p_1, \cdots, p_n \geqslant 1$, $I_p(f) \in H(p_0, p_1, \cdots, p_n)(R)$ and $\langle DI_p(f), h\rangle_H = pI_{p-1}(g)$, where*

$$g(t_1, t_2, \cdots, t_{p-1}) = \int_0^1 f(t_1, t_2, \cdots, t_{p-1}, t_p)h'(t_p)dt_p \quad \text{for } h \in H. \tag{8}$$

As an approximating sequence for $I_p(f)$, we can take $I_p(f_k)$ where f_k is a special step function in the sense of Itô [2] which tends to f in $L^2([0,1]^p)$. In the proof, the following equality (which is a generalization of Theorem 2.2 of [2]) plays an important role.

$$\begin{aligned} &I_p(f)I_q(g) \\ &= \sum_{l=0}^{p\wedge q} \sum_{\substack{\{i_1,\cdots,i_l\}\subset\{1,2,\cdots,p\} \\ \{j_1,\cdots,j_l\}\subset\{1,2,\cdots,q\}}} I_{p+q-2l}(c(i_1,\cdots,i_l; j_1,\cdots,j_l)f\otimes g), \end{aligned} \tag{9}$$

where $\sum_{\substack{\{i_1,\cdots,i_l\}\subset\{1,2,\cdots,p\} \\ \{j_1,\cdots,j_l\}\subset\{1,2,\cdots,q\}}}$ denotes, for each fixed $l \leqslant p \wedge q$, the sum over all possible ways of choosing l different elements $i_1 < i_2 < \cdots < i_l$ from $\{1,2,\cdots,p\}$ and then associating $j_k \in \{1,2,\cdots,q\}$ to i_k $(k=1,2,\cdots,l)$ such that $\{j_1, j_2, \cdots, j_l\}$ are different elements in $\{1,2,\cdots,q\}$, and $c(i_1,\cdots,i_l; j_1,\cdots,j_l)f\otimes g$ is defined by

$$\begin{aligned} &c(i_1,\cdots,i_l; j_1,\cdots,j_l)f\otimes g(t_1,\cdots,\widehat{t_{i_1}},\cdots,\widehat{t_{i_l}},\cdots,t_p, \\ &\qquad\qquad s_1,\cdots,\widehat{s_{j_1}},\cdots,\widehat{s_{j_l}},\cdots,s_q) \\ &= \int_0^1 \cdots \int_0^1 f(t_1,\cdots,t_p)g(s_1,\cdots,s_q)du_1\cdots du_l, \\ &\qquad \begin{matrix} t_{i_1}\to u_1 & s_{j_1}\to u_1 \\ \vdots & \vdots \\ t_{i_l}\to u_l & s_{j_l}\to u_l \end{matrix} \end{aligned} \tag{10}$$

where, for example, $\widehat{t_{i_1}}$ means that the variable t_{i_1} is removed and $t_{i_1} \to u_1$ means that the variable t_{i_1} is replaced by the variable u_1.

From (8) we see that $D^{p+1}I_p(f)=0$ for every $f\in\widehat{L^2}([0,1]^p)$. Also we can prove $LI_p(f)=-pI_p(f)$.

Theorem 3. *Let F be a real-valued Wiener functional given by $F=\sum_{p=0}^n I_p(f_p)$, $f_p\in\widehat{L^2}([0,1]^p)$, $p=1,2,\cdots,n$. If $f_n\neq 0$, then the probability law on R induced by F is absolutely continuous.*

Theorem 4. *Let $F=(F^1,F^2,\cdots,F^n)$ be an R^n-valued Wiener functional given by*

$$F^i=\sum_{p=0}^{N_i} I_p(f_p^{(i)}),\qquad i=1,2,\cdots,n,\ f_p^{(i)}\in\widehat{L^2}([0,1]^p).$$

We assume that F satisfies that there exists $h\in H$ such that

$$\int_0^1\cdots\int_0^1 f_{N_1}^{(1)}(t_1,t_2,\cdots,t_{N_1})h'(t_2)\cdots h'(t_{N_1})dt_2\cdots dt_{N_1}$$

$$\vdots$$

$$\int_0^1\cdots\int_0^1 f_{N_n}^{(n)}(t_1,t_2,\cdots,t_{N_n})h'(t_2)\cdots h'(t_{N_n})dt_2\cdots dt_{N_n}$$

are linearly independent in $L^2[0,1]$. Then the probability law on R^n induced by F is absolutely continuous.

References

[1] L. Gross: Abstract Wiener spaces. Proc. Fifth Berkeley Sympos. Math. Statist. and Prob., Univ. of California Press, vol. 1, part 1, pp. 31–42 (1965).

[2] K. Itô: Multiple Wiener integral. J. Math. Soc. Japan, **3**, 157–167 (1951).

[3] P. Malliavin: Stochastic Calculus of Variation and Hypoelliptic Operators. Proc. Int. Sympos. on S.D.E., Kyoto, Kinokuniya, pp. 195–263 (1978).