54. *Hyperbolic Nonwandering Sets without Dense Periodic Points*

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(Communicated by Kôsaku YOSIDA, M. J. A., Sept. 12, 1978)

Let $f: M\to M$ be a C^∞ diffeomorphism of a closed C^∞ manifold M, and let $\Omega(f)$ be the nonwandering set of f. $\Omega(f)$ is hyperbolic if $\Omega(f)$ is compact and the restriction $T_{\Omega(f)}M$ of the tangent bundle TM of M on $\Omega(f)$ splits into the Whitney sum of Tf-invariant subbundles

$$T_{\Omega(f)}M=E^s\oplus E^u,$$

such that given a Riemannian metric on TM there are positive numbers c and $\lambda<1$ such that $|Tf^n v|<c\lambda^n|v|$, for $v\in E^s$ and $n>0$, and $|Tf^{-n}v|<c\lambda^n|v|$, for $v\in E^u$ and $n>0$. The following problem was suggested in [3].

Problem. If a nonwandering set $\Omega(f)$ is hyperbolic, are the periodic points dense in $\Omega(f)$?

Newhouse and Palis proved that the answer is affirmative when M is a two dimensional closed manifold ([1] and [2]).

In this paper we give the following

Theorem. *Suppose* $\dim M\geq 4$. *Then there is a diffeomorphism* $F: M\to M$ *such that the nonwandering set* $\Omega(F)$ *is hyperbolic but its periodic points are not dense in* $\Omega(F)$.

Construction. To simplify the construction, we assume $\dim M=4$.

1. Denote $D=[-2,6]\times[-1,3]\subset R^2$. Let an embedding $f: D\to D$ satisfy the followings (Fig. 1). Suppose that real numbers $a_{-1},\cdots,a_6$ satisfy

(1.1) $a_{-1}=-2<-1<a_0=-a_1<0<a_1<1<a_2<a_3<a_4<4<a_5<5<a_6=6$, and the rectangle A_i $(i=0,\cdots,6)$ is given by

$$A_i=\{(x,y)\in D\,|\,a_{i-1}\leq x\leq a_i\}.$$

Then f satisfies (1.2)–(1.5).

(1.2) $f|A_0, f|A_2$ and $f|A_6$ are contractions with three sinks $(-1,0)$, $(1,0)$ and $(5,2)$,

(1.3) $f(A_4)\subset \operatorname{int} A_0$,

(1.4) $f|A_i: A_i\to f(A_i)$ $(i=1,3,5)$ maps A_i linearly onto a rectangle $f(A_i)$, expanding horizontally and contracting vertically. There are two hyperbolic fixed points, $(0,0)$ and $(4,2)$.

(1.5) There are numbers $\alpha>1$ and $0<\beta<1$ such that

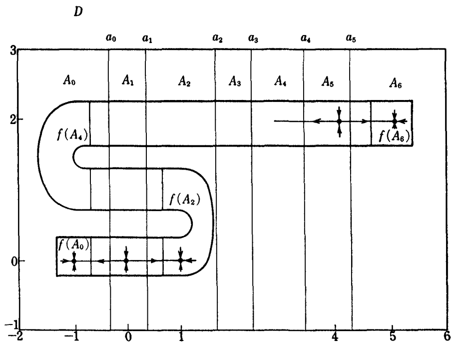

Fig. 1

$$f(x, y)=\begin{cases}(\alpha x, \beta y) & \text{for } (x, y) \in A_1 \\ (\alpha(x-4)+4,\ \beta(y-2)+2) & \text{for } (x, y) \in A_5.\end{cases}$$

2. Let $D' \subset R^2$ satisfy the followings (Fig. 2). D' is a neighbourhood of $(\{0\}\times[-1, 1]) \cup ([-2, 0]\times\{0\})$ which is diffeomorphic to a 2-dimensional disk, and there is a sufficiently small positive number ε

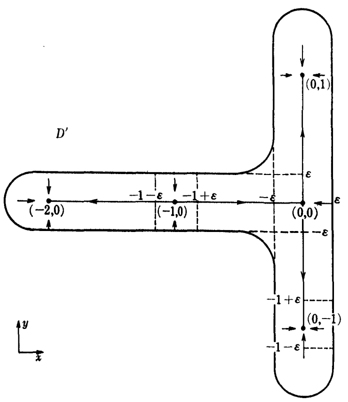

Fig. 2

such that

$$\{(x, y) \in D' \mid |y+1| \le \varepsilon\} = [-\varepsilon, \varepsilon]\times[-1-\varepsilon, -1+\varepsilon]$$

and

$$\{(x, y) \in D' \mid |x+1| \le \varepsilon\} = [-1-\varepsilon, -1+\varepsilon]\times[-\varepsilon, \varepsilon].$$

Let an embedding $g: D' \to D'$ satisfy (2.1)–(2.9).

(2.1) $g(D') \subset \operatorname{int} D'$,

(2.2) g is isotopic to the identity,

(2.3) $\bigcap_{n>0} g^n(D') = (\{0\}\times[-1, 1]) \cup ([-2, 0]\times\{0\})$,

(2.4) There are five fixed points: three sinks $(-2, 0)$, $(0, 1)$, $(0, -1)$, and two saddle points $(0, 0)$, $(-1, 0)$.

(2.5) $W^u((0, 0)) = \{0\}\times(-1, 1)$,

(2.6) $W^u((-1, 0)) = (-2, 0)\times\{0\}$,

(2.7) $W^s((0, 0)) \cap D' = \{(x, 0) \in D' \mid x \ge -1\}$, where $W^s(p)$ (resp. $W^u(p)$) is the stable (resp. unstable) manifold through p. $(-1, 1)$ and $(-2, 0)$ denote open intervals.

(2.8) $g(x, y)=\left(\frac{1}{2}x, \frac{1}{2}(y+1)-1\right)$ if $|y+1|\leq\varepsilon$,

(2.9) $g(x, y)=\left(2(x+1)-1, \frac{1}{2}y\right)$ if $|x+1|\leq\varepsilon$.

3. Define

$$N=D\times D'\bigcup_{\psi} D^3(\delta)\times[0, 1],$$

where

$$D^3(\delta)=\{(y_1, y_2, y_3)\in R^3 \mid \sqrt{y_1^2+y_2^2+y_3^2}\leq\delta\}$$

and

$$0<\delta<\frac{1}{4}\varepsilon.$$

The attaching map

$$\psi: D^3(\delta)\times([0, \varepsilon]\cup[1-\varepsilon, 1])\rightarrow D\times D'$$

is given by

$$\psi(y_1, y_2, y_3, t)=\begin{cases}(y_1, y_2, t, y_3-1) & \text{if } 0\leq t\leq\varepsilon\\ (y_1+4, y_2+2, y_3-1, 1-t) & \text{if } 1-\varepsilon\leq t\leq 1\end{cases}$$

(Fig. 3).

In §§ 4–10, we will construct an embedding $F: N\rightarrow N$. After this, (x_1, x_2, x_3, x_4) (resp. (y_1, y_2, y_3, t)) denotes a point of $D\times D'\subset N$ (resp. $D^3(\delta)\times[0, 1]\subset N$).

4. For $(x_1, x_2, x_3, x_4)\in D\times D'$ with $|x_3+1|\geq\varepsilon$ and $|x_4+1|\geq\varepsilon$, define

(4.1) $F(x_1, x_2, x_3, x_4)=(f(x_1, x_2), g(x_3, x_4))$.

5. For $(x_1, x_2, x_3, x_4)\in D\times D'$ with $\frac{1}{4}\varepsilon\leq|x_4+1|\leq\varepsilon$, define

(5.1) $F(x_1, x_2, x_3, x_4)=(f_{|x_4+1|}(x_1, x_2), g(x_3, x_4))$, where $f_t: D\rightarrow D$ $(0\leq t\leq\varepsilon)$ is an isotopy satisfying (5.2)–(5.6). Suppose that positive numbers $b_1, \cdots, b_4$ satisfy

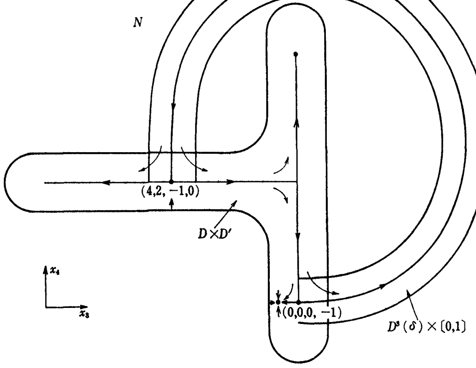

Fig. 3

(5.2) $0<b_1<b_2<\delta<b_3<b_4<a_1, \quad \alpha b_1<b_2,$

and

$$b_4<\min\{4-a_4, a_5-4\}.$$

Then

(5.3) $f_t(x_1, x_2)=f(x_1, x_2)$ if $|x_1|<b_1$ or $|x_1|>b_4$,

(5.4) $f_t=f$ for $\frac{1}{2}\varepsilon\leq t\leq\varepsilon$,

(5.5) $f_t=f_0$ for $0\leq t\leq\frac{1}{4}\varepsilon$,

and

(5.6) $f_t(x_1, x_2)=(\bar{f}_t(x_1), \beta x_2)$ for $|x_1|\leq b_4$,

where $\bar{f}_t$ is an isotopy of a neighbourhood of 0 in R^1 and $\bar{f}_0$ has five fixed points: three sources $0, \pm b_3$, and two sinks $\pm b_2$.

6. For $(x_1, x_2, x_3, x_4)\in D\times D'$ with $|x_4+1|<\frac{1}{4}\varepsilon$, F is defined as follows. Let

(6.1) $U=\{(x_1, x_2, x_3, x_4)\in D\times D' \mid \sqrt{x_1^2+x_2^2+(x_4+1)^2}\leq\delta\}$,

and

(6.2) $U_1=\{(x_1, x_2, x_3, x_4)\in D\times D' \mid \sqrt{x_1^2+x_2^2+(x_4+1)^2}\leq\delta_1\}$,

where $b_2<\delta_1<\delta$.

Then F is defined as follows.

(6.3) $F(x_1, x_2, x_3, x_4)=(f_0(x_1, x_2), g(x_3, x_4))$ if $(x_1, x_2, x_3, x_4)\in D\times D'-U$ and $|x_4+1|<\frac{1}{4}\varepsilon$,

(6.4) $F(x_1, x_2, x_3, x_4)=\left(f_0(x_1, x_2), \bar{g}(x_1, x_2, x_3, x_4), \frac{1}{2}(x_4+1)-1\right)$

if $(x_1, x_2, x_3, x_4)\in U\cap F^{-1}(U)$,

where $\bar{g}$ satisfies (6.5)–(6.7).

(6.5) $\bar{g}(x_1, x_2, x_3, x_4)=\frac{1}{2}x_3$ near the frontier of U,

(6.6) $\bar{g}(x_1, x_2, x_3, x_4)=2x_3$ if $(x_1, x_2, x_3, x_4)\in U_1$ and $-\frac{1}{4}\varepsilon\leq x_3\leq\frac{1}{2}\varepsilon$,

and

(6.7) $\bar{g}(x_1, x_2, x_3, x_4)$ does not depend on x_1 if $|x_1|<b_1$.

(6.8) $F(\{(x_1, x_2, x_3, x_4)\in U \mid x_3<0\})\subset\{(x_1, x_2, x_3, x_4)\in U \mid x_3<0\}$.

In $\{(x_1, x_2, x_3, x_4)\in U \mid x_3<0\}$ there are only a finite number of nonwandering points, which are hyperbolic fixed points. Furthermore F satisfies the conditions in § 10.

7. On $D^3(\delta)\times[0, 1-\varepsilon]$, F is given as follows

(7.1) $F(y_1, y_2, y_3, t)=\left(f_0(y_1, y_2), \frac{1}{2}y_3, \phi(y_1, y_2, y_3, t)\right)\in D^3(\delta)\times[0, 1]$,

where ϕ satisfies the followings.

If $\sqrt{y_1^2+y_2^2+y_3^2}<\delta_1$ or $\frac{1}{2}<t$,

(7.2) $\phi(y_1, y_2, y_3, t)$ depends only on t

and

(7.3) $\frac{\partial\phi}{\partial t}>0.$

(7.4) $\phi(y_1, y_2, y_3, t)=1-\frac{1}{2}(1-t)$ for $1-2\varepsilon\le t\le 1-\varepsilon$.

(7.5) $\phi(y_1, y_2, y_3, t)=\bar{g}(y_1, y_2, t, y_3-1)$ if $0\le t\le\varepsilon$.

Moreover F satisfies § 10.

8. For $(x_1, x_2, x_3, x_4)\in D\times D'$ with $|x_3+1|<\frac{1}{4}\varepsilon$, F is given as follows. Let $h_t: D\to D$ $(0\le t\le\varepsilon)$ be an isotopy such that

(8.1) $h_t=f$ if $\frac{1}{2}\varepsilon\le t\le\varepsilon$,

(8.2) $h_t(x_1, x_2)=f(x_1, x_2)$ if $-2\le x_1\le 4-b_4$ or $4+b_4\le x_1\le 6$,

and

(8.3) $h_t(x_1, x_2)=f(x_1-4, x_2-2)+(4, 2)$ if $|x_1-4|<b_4$.

Then

(8.4) $F(x_1, x_2, x_3, x_4)=\left(h_0(x_1, x_2), \bar{h}(x_1, x_2, x_3, x_4), \frac{1}{2}x_4\right)$,

where $\bar{h}$ satisfies the followings.

(8.5) $\bar{h}(x_1, x_2, x_3, x_4)=\frac{1}{2}(x_3+1)-1$ if $\sqrt{(x_1-4)^2+(x_2-2)^2+(x_3+1)^2}\le\delta$ and $x_4>\frac{2}{3}\varepsilon$,

(8.6) $\bar{h}(x_1, x_2, x_3, x_4)=2(x_3+1)-1$ if $\sqrt{(x_1-4)^2+(x_2-2)^2+(x_3+1)^2}\ge\delta_2$ or $x_4<\frac{1}{3}\varepsilon$,

where $\delta<\delta_2<\frac{1}{4}\varepsilon$.

(8.7) $\bar{h}(x_1, x_2, x_3, x_4)$ does not depend on x_1 if $|x_1-4|<b_1$.

Furthermore F satisfies § 10.

9. For $(x_1, x_2, x_3, x_4)\in D\times D'$ with $\frac{1}{4}\varepsilon\le|x_3+1|<\varepsilon$, define

(9.1) $F(x_1, x_2, x_3, x_4)=\left(h_{|x_3+1|}(x_1, x_2), 2(x_3+1)-1, \frac{1}{2}x_4\right)$.

10. F is an embedding of N such that

(10.1) $F(N)\subset\operatorname{int} N$,

and

(10.2) F is isotopic to the identity.

11. Straightening the corner (and modifying F near the corner), we can regard N as a submanifold of M which is diffeomorphic to $D^3 \times S^1$. Extend F to a diffeomorphism of M such that the nonwandering sets of F in $M-N$ consists of a finite number of hyperbolic fixed points.

12. The nonwandering set of F consists of a finite number of hyperbolic fixed points and two non-periodic orbits $\{(x_1, x_2, 0, 0) \in D \times D' \mid (x_1, x_2)$ satisfies $(12.\ i)\}$ $(i=1,2)$, where

(12.1) there is an integer n_0 such that

$$f^n(x_1, x_2) \in A_5 \quad \text{if } n<n_0,$$
$$f^n(x_1, x_2) \in A_3 \quad \text{if } n=n_0,$$
$$f_n(x_1, x_2) \in A_1 \quad \text{if } n>n_0,$$

and

(12.2) there is an integer n_0 such that

$$f^n(x_1, x_2) \in A_5 \quad \text{if } n<n_0,$$
$$f^n(x_1, x_2) \in A_1 \quad \text{if } n\geq n_0.$$

The details will be published elsewhere.

References

[1] S. Newhouse and J. Palis: Hyperbolic nonwandering sets on two-dimensional manifolds. Dynamical Systems (M. M. Peixoto ed.), Academic Press (1973).

[2] J. Palis and C. C. Pugh: Fifty problems in dynamical systems. Dynamical Systems-Warwick 1974 (A. Manning ed.), Lectures Notes in Math., no. 468, Springer-Verlag (1975).

[3] S. Smale: Differentiable dynamical systems. Bull. Amer. Math. Soc., **73**, 747–817 (1967).