52. *Best Possibility of an Integral Test for Sample Continuity of L_p-Processes ($p\geq 2$)*

By Norio KÔNO
Institute of Mathematics, Yoshida College, Kyoto University

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§ 1. **Introduction.** Let $\{X(t,\omega);\ 0\leq t\leq 1,\ \omega\in\Omega\}$ be a real valued separable stochastic process defined on a probability space $(\Omega,\mathfrak{A},P)$. We concern a best possible integral test for sample continuity of all processes belonging to an indicated class. For this aim, we define for $p\geq 1$,

S_p = a collection of all separable stochastic processes $\{X(t,\omega);\ 0\leq t\leq 1,\ \omega\in\Omega\}$ up to equivalent class such that

$$(E[|X(t)|^p])^{1/p}=\|X(t)\|_p<+\infty \qquad \text{for all } 0\leq t\leq 1,$$

Σ = a collection of all continuous function on [0, 1] which are non-negative, non-decreasing and zero at the origin,

and for $\sigma\in\Sigma$

$$S_p(\sigma)=\{\{X(t)\}\in S_p;\ \|X(t)-X(s)\|_p\leq\sigma(|t-s|)\}.$$

Then the following integral test for sample continuity of all processes belonging to $S_p(\sigma)$ is known ([1]).

Theorem A. *If*

$$I_p(\sigma)=\int_{+0} h^{-(1+1/p)}\sigma(h)dh<+\infty,$$

then all processes belonging to $S_p(\sigma)$ have continuous sample paths with probability 1.

The converse statement is not true in general, but Hahn-Klass [2] have proved the following theorem using a rearrangement of σ in case of $p=2$.

Set $\bar{\sigma}(h)=\inf_{y\geq 1} y\sigma(h/y)$.

Theorem B. *All processes belonging to $S_2(\sigma)$ have continuous sample paths with probability* 1 *if and only if $I_2(\bar{\sigma})$ converges.*

In this paper, we establish some relation concerning about $\bar{\sigma}$ and extend Theorem B to $p\geq 2$ by just the analogous method as them.

§ 2. Set

$$\sigma_*(h)=\sup_{\{X(t)\}\in S_p(\sigma)}\ \sup_{\substack{0\leq s\leq h\\ 0\leq t\leq t+s\leq 1}}\|X(t+s)-X(t)\|_p,$$

and

$$\sigma^*(h)=\text{the largest sub-additive minorant of } \sigma,$$

that is, σ^* is characterized by the following:

(i) $\sigma^*\in\Sigma$,

(ii) $\sigma^* \leq \sigma$,

(iii) $\sigma^*(s+t) \leq \sigma^*(s) + \sigma^*(t)$,

(iv) if σ' satisfies (i), (ii) and (iii), then $\sigma' \leq \sigma^*$.

Lemma 1. $\sigma^* = \sigma_*$.

In fact, obviously we have $\sigma_* \leq \sigma$ and $\sigma_* \in \Sigma$. By the triangular inequality and the definition of σ_*, σ_* satisfies (iii) which implies $\sigma_* \leq \sigma^*$ by (iv). Conversely, choose an arbitral random variable X such that $\|X\|_p = 1$ and set $X(t) = \sigma^*(t)X$, then

$$\|X(t) - X(s)\|_p \leq |\sigma^*(t) - \sigma^*(s)| \leq \sigma^*(|t-s|) \leq \sigma(|t-s|).$$

Therefore we have $\{X(t)\} \in S_p(\sigma)$ and it follows from the definition of σ_* that

$$\sigma^*(t) = \|X(t) - X(o)\|_p \leq \sigma_*(t). \qquad \text{Q.E.D.}$$

Lemma 2. *Hahn-Klass' function $\bar{\sigma}$ has the following properties:*

(i) $0 \leq \bar{\sigma} \leq \sigma$,

(ii) $\bar{\sigma} \in \Sigma$,

(iii) *$x\bar{\sigma}(1/x)$ is continuous, non-decreasing on $[1, +\infty)$,*

(iv) *$\bar{\sigma}$ is sub-additive,*

(v) $\bar{\sigma} \leq \sigma^* = \sigma_* \leq 2\bar{\sigma}$.

Proof. In case of $p=2$, the properties (i)–(iii) have been proved in [2], and one can easily apply their proofs to the general case. To prove (iv), assume $h \geq h' \geq 0$ and $h + h' \leq 1$, then it follows by (iii) that

$$\bar{\sigma}(h+h')/(h+h') \leq \bar{\sigma}(h)/h,$$

and again by (iii) we have

$$\bar{\sigma}(h+h') \leq \bar{\sigma}(h) + h'\bar{\sigma}(h)/h \leq \bar{\sigma}(h) + \bar{\sigma}(h').$$

The first inequality of (v) follows from sub-additivity of $\bar{\sigma}$ and the definition of σ^*. For the second inequality of (v), we notice that

$$\|X(h) - X(o)\|_p \leq \sum_{k=1}^{n} \|X(kh/n) - X((k-1)h/n)\|_p \leq n\sigma(h/n), \quad n = 1, 2, \cdots.$$

Therefore for any y with $n \leq y < n+1$, it follows that

$$y\sigma(h/y) \geq n\sigma(h/(n+1)) \geq (n+1)\sigma(h/(n+1))/2 \geq \|X(h) - X(o)\|_p/2$$

holds for any $\{X(t)\} \in S_p(\sigma)$, which yields $2\bar{\sigma} \geq \sigma_*$.

§3. Now we establish an extension of Theorem B.

Theorem. *When $p \geq 2$, all processes belonging to $S_p(\sigma)$ have continuous sample paths with probability 1 if and only if one of the following three conditions is fulfilled:*

(i) $I_p(\bar{\sigma}) < +\infty$,

(ii) $I_p(\sigma^*) < +\infty$,

(iii) $I_p(\sigma_*) < +\infty$.

Remark. One can easily construct an example such that $I_p(\bar{\sigma}) < +\infty$ but $I_p(\sigma) = +\infty$ by the same way as that of [2], who have given such example in case of $p=2$.

For the proof of Theorem, it is sufficient by virtue of Theorem A and Lemma 2 that we construct a stochastic process $\{X(t)\}$ belonging to $S_p(\sigma)$ which does not have continuous sample paths with probability 1 under the condition $I_p(\bar{\sigma})=+\infty$. For this aim, we will modify the proof of [2]. First we need several lemmas.

Lemma 3 ([4, p. 129]). *Let $\{a_n\}$ be a non-negative, non-increasing sequence with $\lim_{n\to\infty} a_n=0$. Then a Fourier cosine series*

$$g(x)=\sum_{n=1}^{\infty} a_n \cos 2\pi nx, \qquad x\in[0,1]$$

converges uniformly on any compact subset of the open interval $(0,1)$. Moreover, for $p>1$, $g(x)$ belongs to $L_p[0,1]$ (with respect to the Lebesgue measure) if and only if

$$\sum_{n=1}^{\infty} a_n^p n^{p-2}<+\infty.$$

Lemma 4 ([4, p. 109]). *Let $\{c_n\}_{n=-\infty}^{\infty}$ be complex numbers such that $\sum |c_n|^p (|n|+1)^{p-2}$ converges for $p\geq 2$. Then, there exists an f in $L^p[0,1]$ such that*

$$c_n=\int_0^1 f(x)e^{-2\pi inx}dx,$$

and

$$\left(\int_0^1 |f(x)|^p\,dx\right)^{1/p} \leq A_p\left(\sum_{-\infty}^{\infty} |c_n|^p (|n|+1)^{p-2}\right)^{1/p},$$

where, A_p is a constant independent of f or $\{c_n\}$.

Lemma 5. *If $I_p(\bar{\sigma})=+\infty$ $(p>1)$, then $x\bar{\sigma}(1/x)\uparrow+\infty$ as $x\uparrow+\infty$.*

Lemma 6. *If $I_p(\bar{\sigma})=+\infty$ $(p>1)$, then there exists a non-negative random variable Y such that*

(i) *for any $y\geq 1$*

$\|Y\wedge y\|_p\leq 6y\bar{\sigma}(1/y)/\bar{\sigma}(1), \qquad (a\wedge b=\min(a,b))$

(ii) $\int^{+\infty} P(Y>y)^{1/p}y^{1/p-1}dy=+\infty.$

Outline of the proof. Analogously as that of [2], set

$$t_0=1, \qquad t_n=\sup\{x\,;\,x\bar{\sigma}(1/x)\leq 2^n\bar{\sigma}(1)\},$$

then there exists a random variable Y such that

a) $p(Y>t_n)=(2^n/t_n)^p=(\bar{\sigma}(1/t_n)/\bar{\sigma}(1))^p$,

b) $p(Y>y)=p(Y>t_{n+1})$ for $2t_n\leq y<t_{n+1}$,

and

$$p(Y>y)=p(Y>t_n)(2t_n-y)/t_n+p(Y>t_{n+1})(y-t_n)/t_n,$$

for $t_n\leq y<2t_n$.

It is easy to check (i) and (ii) for the above Y.

Lemma 7. *Let $\{a_n,\ n=1,2,\cdots\}$ be a non-negative sequence and set for $p>1$,*

$$g_n^p=\sum_{k=n}^{\infty} a_k^p k^{p-2}.$$

Then there exist positive constants B_p and C_p depending only on p such that

(i) $\sum_{n=1}^{\infty} g_n n^{1/p-1} \geq B_p \sum_{n=1}^{\infty} a_n$,

(ii) *if* a_n *is non-increasing,*

$$\sum_{n=1}^{\infty} g_n n^{1/p-1} \leq C_p \sum_{n=1}^{\infty} a_n \qquad \text{(Boas' inequality)}.$$

In case of $p=2$, one can find a proof in [3]. Lemma 7 is also proved analogously.

Lemma 8. *Set* $b_n = P(Y>n)^{1/p}$, $(p \geq 2)$ *for the random variable* Y *in Lemma* 6. *Then there exists a rearrangement* $\bar{b}_n$ *of* b_n *such that*

(i) $\bar{b}_n \leq b_n$,

(ii) $a_n = (\bar{b}_n^p - \bar{b}_{n+1}^p)^{1/p} n^{2/p-1}$ *is positive, non-increasing,*

(iii) $\sum^{\infty} \bar{b}_n n^{1/p-1} = +\infty$,

(iv) $\sum^{\infty} a_n = +\infty$,

(v) $\sum_{k=1}^{j} a_k^p k^{2p-2} + j^p \sum_{k>j}^{\infty} a_k^p k^{p-2} \leq E[(Y \wedge j)^p]$.

In fact, $\bar{b}_n^p$ is defined as the largest convex minorant of b_n^p, then all conditions (i)–(v) are fulfilled by Lemmas 6 and 7, (ii).

Proof of Theorem. We choose $\{a_n;\ n=1,2,\cdots\}$ in Lemma 8 and set

$$X(t,x) = D_p g(x-t) = D_p \sum_{n=1}^{\infty} a_n \cos 2\pi n(x-t),$$

and

$$D_p = \bar{\sigma}(1)/(A_p \pi \cdot 3 \cdot 2^{3-1/p}).$$

Then, $\{X(t,x);\ 0 \leq t \leq 1\}$ is a stochastic process on the probability space $([0,1], dx)$ and belongs to S_p by the definition of $\{a_n\}$ and by Lemma 3.

Since we have

$$\begin{aligned} c_n &= \int_0^1 (X(t+h)x) - X(t,x))e^{-2\pi i n x} dx \\ &= D_p a_{|n|}(e^{-2\pi i n(t+h)} - e^{-2\pi i n t})/2, \end{aligned}$$

it follows by Lemmas 4 and 8, (v) that for $1/j \leq h < 1/(i-1)$,

$$\begin{aligned} & E[|X(t+h) - X(t)|^p]^{1/p} \\ & \leq A_p \Big(\sum_{n=-\infty}^{\infty} |c_n|^p \, (|n|+1)^{p-2} \Big)^{1/p} \\ & \leq A_p D_p \Big(\pi^p 2^{p-1} h^p \sum_{nh \leq 1} a_n^p n^{2p-2} + 2^{p-1} \sum_{nh>1} a_n^p n^{p-2} \Big)^{1/p} \\ & \leq 2^{2-1/p} \pi A_p D_p (E[(Y \wedge j)^p])^{1/p}/j \\ & \leq 3 \cdot 2^{3-1/p} \pi A_p D_p \bar{\sigma}(1/j)/\bar{\sigma}(1) \\ & \leq \bar{\sigma}(h) \leq \sigma(h). \end{aligned}$$

Therefore $\{X(t)\}$ belongs to $S_p(\sigma)$, having discontinuous (unbounded) sample paths with probability 1 because of $\sum_{n=1}^{\infty} a_n = +\infty$ by Lemma 8, (iv).

References

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