3. *A Relative Cohomology in the Frame of the Derived Category*

By Moto-o KINOSHITA and Jun KOJIMA
Department of Mathematics, School of Science and Engineering, Waseda University

(Communicated by Kôsaku YOSIDA, M. J. A., Jan. 12, 1978)

The purpose of this paper is to prove an existence theorem and, a uniqueness theorem in a sense, for the cohomology theory satisfying the following "Axioms". It is an axiomatic formulation of the cohomology theory used in [1].

Let X and Y be topological spaces and let $f: Y \to X$ be a continuous map. For brevity we call this map a fibration from now on. Let $f: Y \to X$ and $f': Y' \to X'$ be two fibrations. Then $(u, v): f' \to f$ is said to be a morphism if and only if $u: X' \to X$ and $v: Y' \to Y$ are both continuous maps such that $fv = uf'$. As for the definition of composition morphisms we follow the usual one.

Let $S(X)$ be the category of sheaves on X and let $D^+S(X)$ be its derived category. In the below we use the same notation as [2]. Our "Axioms" comprises the following (1), (2) and (3).

Axioms. Let $H(f): D^+S(X) \to D^+S(X)$ be a ∂-functor determined by a fibration $f: Y \to X$, and let $\alpha(u, v): H(f) \to Ru_*H(f')u^{-1}$ be a functorial morphism determined by a morphism of fibrations $(u, v): f' \to f$, and further let $\delta(f, g): Rf_*H(g)f^{-1} \to TH(f)$ be a functorial morphism determined by two fibrations $f: Y \to X$, $g: Z \to Y$.

We set the following axioms for the system $\{H(f), \alpha(u, v), \delta(f, g)\}$:

(1) if $f: Y \to X$ and $g: Z \to Y$ are fibrations, then the diagram

$$H(f)F \xrightarrow{i} H(fg)F \xrightarrow{j} Rf_*H(g)f^{-1}F \xrightarrow{k} TH(f)F$$

is a triangle in $D^+S(X)$, where F is an object of $D^+S(X)$ and i, j and k are the morphisms given by

$$i = \alpha(1_X, g)F, \quad j = \alpha(f, 1_Z)F, \quad k = \delta(f, g)F$$

respectively.

(2) (a) $\alpha((u, v)(u', v')) = (Ru_*\alpha(u', v')u^{-1})\alpha(u, v)$

where $(u, v): f' \to f$ and $(u', v'): f'' \to f'$ are morphisms of fibrations.

(b) $\alpha(1_Y, 1_X) = 1_{H(f)}$

where $f: Y \to X$ is a fibration.

(c) if $(u, v): f' \to f$ and $(v, w): g' \to g$ are given fibrations, the diagram

$$\begin{array}{ccc}
Rf_*H(g)f^{-1} & \xrightarrow{Rf_*\alpha(v,w)f^{-1}} & Rf_*Rv_*H(g')v^{-1}f^{-1} \\
 & & \| \\
 & & Ru_*Rf'_*H(g')f'^{-1}u^{-1} \\
\delta(f,g)\Big\downarrow & & \Big\downarrow Ru_*\delta(f',g')u^{-1} \\
 & & Ru_*TH(f')u^{-1} \\
 & & \| \\
TH(f) & \xrightarrow[T\alpha(u,v)]{} & TRu_*H(f')u^{-1}
\end{array}$$

is commutative.

(3) (a) if f is an open injective fibration, then

$$H(f)=RK(f).$$

(b) if $(u, v): f'\to f$ is a morphism of open injective fibrations, then

$$\alpha(u, v)=\zeta(u_*, K(f'), u^{-1})R\rho(u, v).$$

In (3), first, a sheaf $K(f)F$ on X is defined by

$$K(f)F(U)=\{s\in F(U)\,;\, s\,|\,U\cap fY=0\}$$

for f an open injective fibration and F a sheaf on X. Then $K(f)$ is a functor from $S(X)$ to $S(X)$.

Next, we define $\rho(u, v): K(f)\to u_*K(f')u^{-1}$ so that the diagram

$$\begin{array}{ccc}
K(f) & \xrightarrow{\rho(u,v)} & u_*K(f')u^{-1} \\
\Big\downarrow & & \Big\downarrow \\
1_{S(X)} & \xrightarrow[\rho(u)]{} & u_*u^{-1}
\end{array}$$

is commutative, where $(u, v): f'\to f$ is a morphism of open injective fibrations and

$$\rho(u): 1_{S(X)}\to u_*u^{-1}$$

is the canonical functorial morphism.

Finally

$$\zeta(u_*, K(f'), u^{-1}): R(u_*K(f')u^{-1})\to Ru_*RK(f')u^{-1}$$

is the canonical functorial morphism in the derived category $D^+S(X)$.

The statement of our Axioms is, so far, finished. Then we have the following theorems.

Theorem 1. *There exists a system* $\{H(f), \alpha(u, v), \delta(f, g)\}$ *satisfying the axioms stated above.*

Theorem 2. *Let* $\{H(f), \alpha(u, v), \delta(f, g)\}$ *and* $\{\bar{H}(f), \bar{\alpha}(u, v), \bar{\delta}(f, g)\}$ *be two systems satisfying the axioms. Then there exists a unique isomorphism*

$$\omega(f): H(f)\to\bar{H}(f)$$

with the properties

(i) *if* $(u, v): f'\to f$ *is a morphism of fibrations, then*

$$\bar{\alpha}(u, v)\omega(f)=(Ru_*\omega(f')u^{-1})\alpha(u, v)$$

(ii) *if* f *is an open injective fibration, then*

$$\omega(f)=1_{RK(f)}.$$

Furthermore, assume that

$$\delta(f,\phi\to Y)=\bar{\delta}(f,\phi\to Y)$$

holds for any open injective fibration $f: Y\to X$. *Then for fibrations* $f: Y\to X$ *and* $g: Z\to Y$ *the following diagram is commutative*:

$$\begin{array}{ccc} Rf_*H(g)f^{-1} & \xrightarrow{\delta(f,g)} & TH(f) \\ {\scriptstyle Rf_*\omega(g)f^{-1}}\downarrow & & \downarrow{\scriptstyle T\omega(f)} \\ Rf_*\bar{H}(g)f^{-1} & \xrightarrow[\bar{\delta}(f,g)]{} & T\bar{H}(f) \end{array}$$

Outline of Proof. Let $f: Y\to X$ be an open injective fibration. Then we define a ∂-functor $L(f): C^+S(X)\to C^+S(X)$, by

$$(L(f)F)^n=F^u\oplus f_*f^{-1}F^{n-1},$$

$$d^n_{L(f)F}=\begin{pmatrix} d^n_F & 0 \\ (-1)^n\rho(f)F^n & f_*f^{-1}d^{n-1}_F \end{pmatrix}$$

for an object $F=(F^n)$ of $C^+S(X)$. Let $\tilde{H}(f)=RL(f): D^+S(X)\to D^+S(X)$ be the derived functor of $L(f)$. For $f: Y\to X$ fibration, let M be the mapping cylinder of f, and let $j: Y\to M, p: M\to X$ be the canonical maps, as defined in [3]. Then we put

$$H(f)=p_*\tilde{H}(j)p^{-1}$$

where we note that the functor $p_*: S(M)\to S(X)$ is exact. Theorem 2 follows from the exactness of the functor p_*.

Our thanks are due to [1], from which our formulation of axioms and theorems are inspired.

References

[1] M. Sato, T. Kawai, and M. Kashiwara: Microfunctions and pseudodifferential equations. Lecture Notes in Math., **287**, Springer (1973).

[2] R. Hartshorn: Residue and duality. Lecture Notes in Math., **20**, Springer (1966).

[3] H. Komatsu: Cohomology of morphisms of sheafed spaces. J. Fac. Sci. Univ. Tokyo, Sect. IA, **18**, 287–327 (1971).