1. *Studies on Holonomic Quantum Fields.* VI

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In this series of papers [1] ~ [3] it has been realized that a deep connection exists between the deformation theory of linear differential equations and field operators belonging to the Clifford group. The aim of the present article is to study the Riemann-Hilbert problem on the complex sphere P_C^1 [5], [6] from the above standpoint. In the case where the branch points $a_1, \cdots, a_n, \infty$ all lie on the real line P_R^1, we shall show in § 2 the equivalence of (i) finding a multi-valued analytic function with a prescribed monodromy property, and (ii) constructing a field operator which induces a specified rotation. We then give a canonical scheme of the latter.

We follow the same notation as in [1] ~ [3] unless otherwise stated explicitly.

1. Let W_1, W_2 be orthogonal vector spaces equipped with the inner product $\langle\ ,\ \rangle_{W_1}, \langle\ ,\ \rangle_{W_2}$. Their tensor product $W=W_1\otimes W_2$ is naturally endowed with an orthogonal structure by setting $\langle w_1\otimes w_2, w_1'\otimes w_2'\rangle_W =\langle w_1, w_1'\rangle_{W_1}\cdot\langle w_2, w_2'\rangle_{W_2}$ $(w_1, w_1'\in W_1, w_2, w_2'\in W_2)$. We denote by ι (resp. ι_ν) the element of $\mathrm{Hom}_C(W, W^*)$ (resp. $\mathrm{Hom}_C(W_\nu, W_\nu^*)$) which defines the inner product $\langle\ ,\ \rangle_W$ (resp. $\langle\ ,\ \rangle_{W_\nu}$), i.e. $\iota(w)(w')=\langle w, w'\rangle_W$, $\iota_\nu(w_\nu)(w_\nu')=\langle w_\nu, w_\nu'\rangle_{W_\nu}$ $(\nu=1,2)$. Also a κ-norm on $A(W_1)$ induces one on $A(W)$; namely let $\kappa_1\in\mathrm{Hom}_C(W_1, W_1^*)$ be an element such that $\kappa_1+{}^t\kappa_1=\iota_1$. Then $\kappa=\kappa_1\otimes\iota_2\in\mathrm{Hom}_C(W, W^*)$ clearly satisfies $\kappa+{}^t\kappa=\iota$ [3], [4].

Now let $W_1=\{w(x)\}$ be the orthogonal space of functions on $\boldsymbol{R}^1$ equipped with the inner product $\langle w, w'\rangle_{W_1}=\int_{-\infty}^{+\infty} dx w(x)w'(x)=\langle w', w\rangle_{W_1}$. Let $\psi(x)=\int_{-\infty}^{+\infty}\underline{du}\sqrt{0+iu}\, e^{ixu}\psi(u)$ be the free fermion operator in one dimensional space, where $\psi(u)^\dagger=\psi(-u)$, $\psi(u)$ $(u>0)$ denotes creation and annihilation operator, respectively [1]. By identifying $w\in W_1$ with $\int_{-\infty}^{+\infty} dx w(x)\psi(x)$ we regard $\psi(x)$ as an element of $W_1\subset\overline{G}(W_1)$.

Denoting by κ_1 the element of $\mathrm{Hom}_C(W_1, W_1^*)$ corresponding to the holonomic decomposition into the above creation and annihilation operators, we have $\langle\psi(x)\psi(x')\rangle_{\kappa_1}=\dfrac{1}{2\pi}\dfrac{i}{x-x'+i0}$, $\langle\psi(x)\psi(x')\rangle_{{}^t\kappa_1}=\dfrac{1}{2\pi}\dfrac{-i}{x-x'-i0}$ and $[\psi(x), \psi(x')]_+=\langle\psi(x), \psi(x')\rangle_{W_1}=\delta(x-x')$. Notice that $\iota_1^{-1}{}^t\kappa_1$ and

$\iota_1^{-1}\kappa_1$ are projection operators onto the space of boundary values of holomorphic functions on the upper and lower half plane, respectively.

As W_2 we take C^m and choose a basis e_i such that $\langle e_i, e_j\rangle_{W_2}=\delta_{ij}$ $(i,j=1,\cdots,m)$. For $w\in W_1$ we set $w^{(i)}=w\otimes e_i$. In what follows the norm Nr and the vacuum expectation value $\langle\ \rangle$ on $A(W)=A(W_1\otimes W_2)$ shall refer to $\kappa=\kappa_1\otimes\iota_2$ explained above.

2. Let P_C^1 denote the complex projective line. We fix a coordinate and set $P_C^1-\{\infty\}=D_+\cup R^1\cup D_-$, $D_\pm=\{\mathrm{Im}\,x\gtrless 0\}$. Suppose we are given n points $a_1<\cdots<a_n$ on R^1 and n matrices $M_1,\cdots,M_n\in \mathrm{GL}(m,C)$ arbitrarily. The Riemann-Hilbert problem amounts to finding $m\times m$ matrices $Y_\pm(x)$ of holomorphic functions on $D_\pm$, respectively, with the properties (i) $Y_\pm(x)$ has at most regular singularities at $a_1,\cdots,a_n,\infty$, and (ii) their boundary values are related through

(1) $$Y_-(x-i0)=Y_+(x+i0)M(x),\qquad x\in R^1-\{a_1,\cdots,a_n\}$$

where $M(x)=(m_{ij}(x))=M_\nu M_{\nu+1}\cdots M_n$ for $a_{\nu-1}<x<a_\nu$ $(\nu=1,\cdots,n,\ a_0=-\infty)$, $=1$ for $a_n<x$.

First assume $M_\nu\in O(m,C)$ $(\nu=1,\cdots,n)$ and consider the rotation T in $W=W_1\otimes W_2$ given by

(2) $$(Tw^{(j)})(x)=\sum_{i=1}^m w^{(i)}(x)m_{ij}(x),\qquad w\in W,\ j=1,\cdots,m.$$

Suppose that T be induced by an even element $\varphi\in\overline{G}(W)$ of the form

(3) $$\mathrm{Nr}\,(\varphi)=\exp(\rho/2)$$
$$\rho=\sum_{i,j=1}^m\iint_{-\infty}^{+\infty}dx\,dx'\,r_{ij}(x,x')\psi^{(i)}(x)\psi^{(j)}(x').$$

In other words we assume the following commutation relation with ψ's:

(4) $$\varphi\psi^{(j)}(x)=\sum_{i=1}^m\psi^{(i)}(x)\varphi m_{ij}(x),\qquad j=1,\cdots,m.$$

For $i,j=1,\cdots,m$ and $x_0>a_n$ we set

(5) $$y_{+ij}(x_0;x)=-2\pi i(x_0-x)\langle\psi^{(i)}(x_0)\psi^{(j)}(x)\varphi\rangle$$
$$y_{-ij}(x_0;x)=-2\pi i(x_0-x)\langle\psi^{(i)}(x_0)\varphi\psi^{(j)}(x)\rangle.$$

Applying (23)~(26) in [2] we have

(6) $$y_{\pm ij}(x_0;x)=\delta_{ij}+2\pi i(x_0-x)\iint_{-\infty}^{+\infty}dx_1dx_2\frac{1}{2\pi}\frac{i}{x_0-x_1+i0}\frac{1}{2\pi}\frac{i}{x-x_2\pm i0}r_{ij}(x_1,x_2)$$

which implies that as a function of x $Y_\pm(x_0;x)=(y_{\pm ij}(x_0;x))$ is analytically prolongable to $D_\pm$, respectively, and $Y_\pm|_{x=x_0}=1$. Moreover from (4) we see that they also satisfy (1).

Conversely we may construct an operator φ satisfying (4) once we know matrices $Y'_\pm(x)=(y'_{\pm ij}(x))$ of holomorphic functions on $D_\pm$ with the monodromy property (1). Recall that the condition $T_\varphi=T$ is equivalently stated in terms of $\rho\in\Lambda^2(W)$ as [3], [4]

(7) $$\rho(\kappa T+{}^t\kappa)=T-1$$

where we have used the identification $\Lambda^2(W)\subset W\otimes W\cong\mathrm{Hom}_C(W^*,W)$.

Lemma. *Let $\chi_\pm \in \mathrm{Hom}_C(W, W)$ be invertible operators such that*

(8) $$\iota^{-1}\kappa\chi_+\iota^{-1t}\kappa=0, \quad \iota^{-1t}\kappa\chi_-\iota^{-1}\kappa=0, \quad \chi_-=\chi_+T.$$

Then $\rho'=(\chi_+^{-1}-\chi_-^{-1})(\iota^{-1}\kappa\chi_+ + \iota^{-1t}\kappa\chi_-)\iota^{-1} \in \mathrm{Hom}_C(W^*, W)$ *satisfies* (7). *If in addition*

(8)′ $$\iota^{-1}\kappa^t\chi_+^{-1}\iota^{-1t}\kappa=0, \qquad \iota^{-1t}\kappa^t\chi_-^{-1}\iota^{-1}\kappa=0,$$

then $-{}^t\rho'$, *and hence* $\rho=\frac{1}{2}(\rho'-{}^t\rho') \in \Lambda^2(W)$, *satisfies* (7).

If we take as $\chi_\pm$ the multiplication operators $w^{(j)}(x)\mapsto\sum_{i=1}^m w^{(i)}(x)\cdot y'_{\pm ij}(x)$, it is easy to see that all the conditions (8), (8)′ are fulfilled. Notice that such $Y'_\pm(x)$ and φ are not uniquely determined by the rotation T.

In the case where the monodromy matrices $M_\nu \in \mathrm{GL}(m, C)$ are not necessarily orthogonal, we let $W_2=C^{2m}$ and choose a basis e_i, e_i^* such that $\langle e_i, e_j\rangle=0$, $\langle e_i^*, e_j^*\rangle=0$ and $\langle e_i, e_j^*\rangle=\delta_{ij}$ $(i, j=1, \cdots, m)$. Set $w^{(i)}=w\otimes e_i$, $w^{*(i)}=w\otimes e_i^*$ $(w \in W_1)$. Then $\tilde{M}_\nu=\begin{pmatrix} M_\nu & \\ & {}^tM_\nu^{-1}\end{pmatrix} \in \mathrm{SO}(W_2)$, and

(9) $$(Tw^{(j)})(x)=\sum_{i=1}^m w^{(i)}(x)m_{ij}(x), \qquad (Tw^{*(j)})(x)=\sum_{i=1}^m w^{*(i)}(x)m'_{ij}(x)$$

defines a rotation in $W=W_1\otimes W_2$, where $(m'_{ij}(x))={}^tM(x)^{-1}$. Hence the general case is reduced to the case of orthogonal monodromy of double size. In particular we note that

(10) $$\begin{aligned} y_{+ij}(x_0; x)&=-2\pi i(x_0-x)\langle\psi^{*(i)}(x_0)\psi^{(j)}(x)\varphi\rangle \\ y_{-ij}(x_0; x)&=-2\pi i(x_0-x)\langle\psi^{*(i)}(x_0)\varphi\psi^{(j)}(x)\rangle \end{aligned}$$

give functions with the correct monodromy property (1).

3. In the case $n=1$ the Riemann-Hilbert problem admits elementary solutions; namely we may take $Y_\pm(x)=(x-a\pm i0)^{-L}$, where L is an $m\times m$ matrix satisfying $e^{2\pi iL}=M$. From the lemma, the corresponding operator $\varphi=\varphi(a; L)$, subject to the normalization $\langle\varphi\rangle=1$, is explicitly given by

(11) $$\mathrm{Nr}(\varphi(a; L))=\exp(\rho(a; L)/2)$$

where

$$\frac{1}{2}\rho(a; L)=\sum_{i,j=1}^m \iint_{-\infty}^{+\infty} dxdx' r_{ij}(x-a, x'-a; L)\psi^{(i)}(x)\psi^{*(j)}(x')$$
$$=\sum_{i,j=1}^m \iint_{-\infty}^{+\infty} \underline{du}\,\underline{du'}\tilde{r}_{ij}(u, u'; L)e^{ia(u+u')}\psi^{(i)}(u)\psi^{*(j)}(u'),$$

(12) $$\begin{aligned} R(x, x'; L)&=(r_{ij}(x, x'; L)) \\ &=((x+i0)^L-(x-i0)^L)\Big(\frac{1}{2\pi}\frac{i}{x-x'+i0}(x'-i0)^{-L} \\ &\qquad +\frac{1}{2\pi}\frac{-i}{x-x'-i0}(x'+i0)^{-L}\Big), \end{aligned}$$

$$\begin{aligned} \tilde{R}(u, u'; L)&=(\tilde{r}_{ij}(u, u'; L)) \\ &=-2\sin\pi L\cdot(u-i0)^{-L+1/2}(u'-i0)^{L+1/2}\frac{-i}{u+u'-i0}. \end{aligned}$$

Now we proceed to construction of a canonical operator which induces the rotation (9). Choose L_ν so that $e^{2\pi i L_\nu}=M_\nu$ $(\nu=1,\cdots,n)$ and set

$$\varphi=\langle\varphi(a_1;L_1)\cdots\varphi(a_n;L_n)\rangle^{-1}\varphi(a_1;L_1)\cdots\varphi(a_n;L_n). \tag{13}$$

Applying the product formula (3.7) in [3] we see that its norm takes the form

$$\begin{gathered}\mathrm{Nr}\,(\varphi)=\exp(\rho/2)\\ \frac{1}{2}\rho=\sum_{\mu,\nu=1}^{n}\sum_{i,j=1}^{m}\iint_{-\infty}^{+\infty}dxdx'\hat{r}_{\mu\nu,ij}(x,x')\psi^{(i)}(x)\psi^{*(j)}(x').\end{gathered} \tag{14}$$

Here $\hat{R}_{\mu\nu}(x,x')=(\hat{r}_{\mu\nu,ij}(x,x'))$ denotes the (μ,ν)-th block of $mn\times mn$ matrix

$$\hat{R}(x,x')=\int_{-\infty}^{+\infty}dx_1(1-RA)^{-1}(x,x_1)R(x_1,x') \tag{15}$$

where

$$\begin{aligned}(1-RA)^{-1}(x,x')=\delta(x-x')\cdot 1\\ +\sum_{l=1}^{\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}dx_1\cdots dx_{2l-1}R(x,x_1)A(x_1,x_2)\\ \cdots R(x_{2l-2},x_{2l-1})A(x_{2l-1},x')\end{aligned} \tag{16}$$

$$\begin{gathered}R(x,x')=(\delta_{\mu\nu}R_\nu(x-a_\nu,x'-a_\nu;L_\nu))\\ A(x,x')=(A_{\mu\nu}(x,x')),\quad A_{\mu\nu}(x,x')=\begin{cases}\dfrac{1}{2\pi}\dfrac{i}{x-x'\pm i0} & (\mu\lessgtr\nu)\\ 0 & (\mu=\nu)\end{cases}.\end{gathered} \tag{17}$$

The infinite series (16) is convergent for sufficiently small $|L_\nu|=(\sum_{i,j}|l_{\nu,ij}|^2)^{1/2}$ $(\nu=1,\cdots,n)$. It is clear that $T_\varphi=T_{\varphi(a_1;L_1)}\cdots T_{\varphi(a_n;L_n)}=T$.

Also we note that from (1), (4), (12) in [4] the logarithmic derivative of the τ-function $\tau_n(a_1,\cdots,a_n)=\langle\varphi(a_1;L_1)\cdots\varphi(a_n;L_n)\rangle$ is given by

$$\begin{aligned}d\log\tau_n(a_1,\cdots,a_n)=\frac{2i}{\pi}\sum_{\mu,\nu=1}^{n}\iiint dx_1dx_2dx_3\,\mathrm{trace}\,(L_\mu\sin^2\pi L_\mu\\ \times(x_1-a_\mu)_-^{-L_\mu-1}(1-AR)^{-1}_{\mu\nu}(x_1,x_2)A_{\nu\mu}(x_2,x_3)(x_3-a_\mu)_-^{L_\mu-1}da_\mu),\end{aligned} \tag{18}$$

where $x_-^L=0$ $(x>0)$, $=|x|^L$ $(x<0)$.

4. The local behavior of $Y_\pm(x_0;x)$ defined in (10) are known from (12), (15) and (16). For $x_0,x\in\boldsymbol{C}-(-\infty,a_n]$ set

$$\begin{aligned}&Y(x_0;x)\\ &=1-2\pi i(x_0-x)\sum_{\mu,\nu=1}^{n}\iint_{-\infty}^{+\infty}dx_1dx_2\frac{1}{2\pi}\frac{i}{x_0-x_1}\hat{R}_{\mu\nu}(x_1,x_2)\frac{1}{2\pi}\frac{i}{x_2-x}.\end{aligned} \tag{19}$$

We have then $Y_\pm(x_0;x)=Y(x_0+i0;x\pm i0)$. Making use of (12) we find that in a neighborhood of $x=a_\nu$

$$Y_+(x_0;x)=\Phi_\nu(x_0;x)(x-a_\nu+i0)^{-L_\nu} \tag{20}$$

where

$\Phi_\nu(x_0;x)$

$$(21)\qquad =2\pi i(x_0-x)\sum_{\mu=1}^{n}\iint_{-\infty}^{+\infty}dx_1dx_2\frac{1}{2\pi}\frac{i}{x_0-x_1+i0}(1-RA)^{-1}_{\mu\nu}(x_1,x_2)$$
$$\times\left((x_2-a_\nu+i0)^{L_\nu}\frac{1}{2\pi}\frac{i}{x_2-x+i0}+(x_2-a_\nu-i0)^{L_\nu}\frac{1}{2\pi}\frac{-i}{x_2-x-i0}\right)$$

is prolongable to a single-valued holomorphic matrix at $x=a_\nu$. On the other hand, we have for any $\theta_0<\theta_1$

$$(22)\qquad |Y(x_0;x)|=O(\sqrt{|x|})\qquad (|x|\to\infty,\ \theta_0<\arg x<\theta_1).$$

The monodromy property (1) and (22) guarantee that $x=\infty$ is also a regular singularity of $Y(x_0;x)$. Hence we may write as

$$(23)\qquad Y(x_0;x)=\Phi_\infty(x_0;x)\cdot x^{L_\infty}$$

with $\Phi_\infty(x_0;x)$ a single-valued holomorphic matrix at $x=\infty$. Set $y(x)=\det Y(x_0;x)\prod_{\nu=1}^{n}(x-a_\nu)^{\mathrm{trace}\,L_\nu}$. Then $y(x)$ is single-valued and holomorphic everywhere in the finite x-plane. At $x=\infty$ it behaves like $f(x)\cdot x^{\mathrm{trace}\,(L_\infty+\sum_{\nu=1}^{n}L_\nu)}$, where $f(x)$ denotes a holomorphic function at $x=\infty$. Since $y(x)$ is a polynomial, trace $(L_\infty+\sum_{\nu=1}^{n}L_\nu)$ is a non-negative integer. From this and (22) it follows that, for sufficiently small $|L_\nu|$ $(\nu=1,\cdots,n)$, L_∞ coincides with $(2\pi i)^{-1}\log M_\infty$, where $M_\infty M_1M_2\cdots M_n=1$ and the branch of log is chosen so that $\log 1=0$. We thus conclude trace $(L_\infty+\sum_{\nu=1}^{n}L_\nu)=0$, and $y(x)=y(x_0)$ is a non-zero constant. In particular $\det\Phi_\nu(x_0;a_\nu)\neq 0$ $(\nu=1,\cdots,n)$, $\det\Phi_\infty(x_0;\infty)\neq 0$.

Summing up, $Y(x_0;x)$ is a solution to the Riemann-Hilbert problem such that it has pre-assigned exponents L_ν at $x=a_\nu$ $(\nu=1,\cdots,n)$, $\det Y\neq 0$ for $x\neq a_1,\cdots,a_n,\infty$ and $Y|_{x=x_0}=1$. It then follows [5][6] that $Y(x_0;x)$ satisfies a Fuchsian system of linear differential equations

$$(24)\qquad \frac{dY}{dx}=\left(\frac{A_1}{x-a_1}+\cdots+\frac{A_n}{x-a_n}\right)Y$$

where

$$(25)\quad A_\nu=A_\nu(x_0;a_1,\cdots,a_n)=-\Phi_\nu(x_0;a_\nu)L_\nu\Phi_\nu(x_0;a_\nu)^{-1}\qquad(\nu=1,\cdots,n)$$

are $m\times m$ matrices independent of x. Remark that since the monodromy representation of (24) is independent of $a_1,\cdots,a_n$, A_ν should satisfy, for fixed x_0, the Schlesinger's equations [7]:

$$(26)\qquad dA_\mu=-\sum_{\nu(\neq\mu)}[A_\mu,A_\nu]\cdot\left(\frac{d(a_\mu-a_\nu)}{a_\mu-a_\nu}+\frac{da_\nu}{x_0-a_\nu}\right)\qquad(\mu=1,\cdots,n).$$

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