28. *Twisted Trace Formula of the Brandt Matrix*

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0. In this note we present some relations between the traces of the Brandt matrices or Hecke operators of different quaternion algebras over different number fields. This is an algebraic analogue of the result of H. Saito [1].

1. Let F be a totally real, cyclic extension of $\boldsymbol{Q}$ of prime degree l, of which the conductor is a prime q different from l. We assume that F has a unit of any signature distribution. (If $l=2$, this is automatically satisfied.) Let B denote a quaternion algebra over F, obtained from one over $\boldsymbol{Q}$ by extending the base field, so that the discriminant, say, D, is a rational integer. Taking, once and for all, a generator of $\mathrm{Gal}(F/\boldsymbol{Q})$, we consider to extend it to an automorphism of B. There are infinitely many such extensions. But we have:

Proposition 1. (i) *If $l \geqq 3$, there is a canonical Galois theoretical 1-1 correspondence between the extension σ such that $\sigma^l=\mathrm{id}$, and subalgebra of B which is a quaternion algebra over $\boldsymbol{Q}$ with discriminant D. Every such σ keeps stable some maximal order of B.*

(ii) *For $l=2$, an extension σ with $\sigma^2=\mathrm{id}$ corresponds to a subalgebra B_0 which is a quaternion algebra over $\boldsymbol{Q}$ (with various discriminant). In order that σ keeps stable some maximal order of B, it is necessary and sufficient that B_0 has discriminant D or Dq.*

2. Throughout the following, B is assumed to be totally definite. We take and fix an extension σ of a generator of $\mathrm{Gal}(F/\boldsymbol{Q})$, satisfying the conditions of Proposition 1, and a maximal order $\mathcal{O}$ of B, stable under the action of σ: ${}^{\sigma}\mathcal{O}=\mathcal{O}$. We can naturally extend σ to the idèle group $B_A^\times$ of B, and denote it again by σ. Note that the archimedean part of $B_A^\times$ is

$$B_\infty^\times=\boldsymbol{H}^\times\times\cdots\times\boldsymbol{H}^\times\ (l\text{ copies})=R_+^l\times SU(2)\times\cdots\times SU(2)$$

where $\boldsymbol{H}$ is the Hamilton quaternion algebra. Let χ denote the representation of $B_A^\times$, defined through the composite of the projection $B_A^\times\to B_\infty^\times$, the projection $B_\infty^\times\to SU(2)\times\cdots\times SU(2)$, and the representation $\rho_k\otimes\cdots\otimes\rho_k$ of $SU(2)\times\cdots\times SU(2)$, where ρ_k is the k-th symmetric tensor representation of $SU(2)$. Then we define M_k, the space of all functions $f(x)$ on $B_A^\times$, taking values in the representation space $\mathcal{F}$ of χ, and satisfying $f(uxa)=\chi(u)f(x)$ for all $u\in U$, $x\in B_A^\times$, $a\in B^\times$, where $U=\prod \mathcal{O}_\mathfrak{p}^\times\times B_\infty^\times$ is the unit group of $B_A^\times$ with respect to $\mathcal{O}$. For the pair $(U, B_A^\times)$, we also have the Hecke algebra $R=R(U, B_A^x)$, and its representation $\mathfrak{T}$ in M_k. (For details, we refer to [3].)

Now we can define a twisting operator T_σ on M_k as follows: $\chi({}^\sigma x)$ and $\chi(x)$ being equivalent representations, we can find an intertwining operator $V_\sigma \in GL(\mathcal{F})$ such that $\chi({}^\sigma x)=V_\sigma^{-1}\chi(x)V_\sigma$, $V_\sigma^l=\mathrm{id}$ for all $x \in B_A^x$. Then for $f \in M_k$, we put

$$(T_\sigma f)(x)=V_\sigma f({}^\sigma x), \qquad x \in B_A^x.$$

It is easy to see that M_k is stable under T_σ, and that $T_\sigma^l=\mathrm{id}$. Moreover, we have, for $e \in R$, $T_\sigma^{-1}\mathfrak{T}(e)T_\sigma=\mathfrak{T}({}^\sigma e)$ where σ is extended naturally to R. In view of the recent result of T. Takagi [4], we can easily prove the following

Proposition 2. *Let SM_k be the subspace of M_k, consisting of all $f(x)$ which satisfies $\mathfrak{T}(e)f=\mathfrak{T}({}^\sigma e)f$ for every $e \in R$.*

(i) *SM_k is stable under T_σ, and has the basis consisting of the common eigen functions $f(x)$ for all $\mathfrak{T}(e)$, $e \in R$.*

(ii) *If $f(x)$ is as in* (i), *we have*

$$T_\sigma f=z\cdot f$$

where z is an l-th root of unity.

Therefore, if we denote $SM_k(z)$ the eigenspace of SM_k for the eigenvalue z of T_σ, we get

$$\text{trace}\,\mathfrak{T}(e)T_\sigma=\sum_z z\cdot\{\text{trace}\,\mathfrak{T}(e)\,|\,SM_k(z)\} \tag{1}$$

where z runs through all l-th roots of unity. This is called the "*Twisted Trace Formula*".

3. To give an explicit formula of (1), we combine the methods of [3], to those of [1]. Let B_A^x decompose into disjoint union of double cosets

$$B_A^x=\bigcup_{i=1}^{H} Ux_iB^x. \tag{2}$$

We set $\Gamma_i=B^x \cap x_i^{-1}Ux_i$, and denote by $\mathcal{F}_i$ the subspace of $\mathcal{F}$ consisting of $v \in \mathcal{F}$ such that $\chi(\gamma)v=v$ for all $\gamma \in \Gamma_i$. Then it is easily verified that the mapping $f \to (f_1, \cdots, f_H)$ gives an isomorphism of M_k onto $\mathcal{F}_1\times\cdots\times\mathcal{F}_H$, where we write $f_i=\chi(x_i^{-1})f(x_i)$. Under this isomorphism, $\mathfrak{T}(e)$ is regarded as an endomorphism of $\mathcal{F}_1\times\cdots\times\mathcal{F}_H$, and devided into H^2 blocks

$$\mathfrak{T}(e)=\begin{pmatrix}\mathfrak{T}_{11}(e) & \cdots & \mathfrak{T}_{1H}(e)\\ & \cdots\cdots\cdots & \\ \mathfrak{T}_{H1}(e) & \cdots & \mathfrak{T}_{HH}(e)\end{pmatrix},$$

$\mathfrak{T}_{ij}(e)$ being a linear mapping of $\mathcal{F}_i$ to $\mathcal{F}_j$. For $e=T(\mathfrak{a})$, $\mathfrak{a}$ being an integral ideal of F, this form of $\mathfrak{T}(e)$ is nothing but the so called "*Brandt Matrix*" (cf. [2], [3]). On the other hand σ induces the permutation of double cosets in (2), and we can write:

$${}^\sigma x_i=u_ix_{\sigma(i)}a_i, \qquad u_i \in U,\ a_i \in B^x.$$

Lemma 1. *If we regard T_σ as an endomorphism of $\mathfrak{F}_1 \times \cdots \times \mathfrak{F}_H$, we have $T_\sigma = (T_{\sigma,ij})$, where*

$$T_{\sigma,ij} = \begin{cases} 0 & \cdots i \neq \sigma(j) \\ V_\sigma \chi(a_j^{-1}) \,|\, \mathfrak{F}_i & \cdots i = \sigma(j). \end{cases}$$

Combining the above Lemma 1 to Lemma 2.1 in [3], we start to evaluate trace $\mathfrak{T}(e)T_\sigma$, and after somewhat lengthy calculation we are led to a very explicit formula, so that we can compare it to the traces of the Hecke operators over $\boldsymbol{Q}$. Let λ be the homomorphism of R to $R_0 = R(\mathfrak{U}_{\boldsymbol{Q}}, GL_2(\boldsymbol{Q}_A))$ defined in [1]. Then we have:

Theorem 1. *Let $\chi_2, \cdots, \chi_l$ be the Dirichlet characters defined* $\bmod Dq$, *of conductor q, and of order l. Let $S_k^0(\Gamma_0(N), \chi_i)$ be the essential part of the space of cusp forms of type (k, N, χ_i). For an integral ideal $\mathfrak{a}$ of F such that $(\mathfrak{a}, Dq) = 1$, let $e = T(\mathfrak{a})$ be an element of R.*

(i) *If $l \geqq 3$, we have*

$$\operatorname{tr} \mathfrak{T}(e)T_\sigma = (\deg \mathfrak{T}(e)) + \operatorname{tr} \mathfrak{T}(\lambda(e)) \,|\, S_{k+2}^0(\Gamma_0(D), 1) + \begin{cases} \dfrac{1}{2} \displaystyle\sum_{i=2}^{l} \operatorname{tr} \mathfrak{T}(\lambda(e)) \,|\, S_{k+2}^0(\Gamma_0(Dq), \chi_i) & \cdots \text{ if } q \nmid D \\ 0 & \cdots \text{ if } q \mid D. \end{cases}$$

(ii) *For $l = 2$,*

$$\operatorname{tr} \mathfrak{T}(e)T_\sigma = (\deg \mathfrak{T}(e)) + \frac{1}{2} \operatorname{tr} \mathfrak{T}(\lambda(e)) \,|\, S_{k+2}^0(\Gamma_0(Dq), \chi_2) + (-1)^{|D|-1} \operatorname{tr} \mathfrak{T}(\lambda(e)) \,|\, S_{k+2}^0(\Gamma_0(D), 1)$$

where, the term $(\deg \mathfrak{T}(e))$ *appears if and only if $k = 0$, and $|D|$ is the number of primes dividing D.*

Taking into account the multiplicity one theorem (cf. [4]), we deduce from this formula and (1), that $SM_k(z) \neq \{0\}$ only for $z = 1$, except for the case $l = 2$, $z = -1$. Moreover, in view of the well known result of [3], we get:

Corollary. *The space of Hilbert modular cusp forms of weight $k+2$ and level D associated to F contains a subspace isomorphic, as* $R' = \boldsymbol{Z}[T(\mathfrak{a}), T(\mathfrak{a}, \mathfrak{a}); (\mathfrak{a}, Dq) = 1]$*-module, to $S_{k+2}^0(\Gamma_0(D), 1)$. It also contains a subspace S such that $S \oplus S$ is isomorphic, as R'-module, to* $\sum_{i=2}^{l} S_{k+2}^0(\Gamma_0(Dq), \chi_i)$ *if $q \nmid D$.*

For $D > 1$, these are new types of the so called "*Doi-Naganuma correspondence*". We remark also that, $\operatorname{tr} \mathfrak{T}(\lambda(e)) \,|\, S_{k+2}^0(\Gamma_0(D), 1) + (\deg \mathfrak{T}(e))$ is equal to the trace of the Brandt matrix of the quaternion algebra over $\boldsymbol{Q}$ with discriminant D (cf. [2]), if $|D|$ is odd. This suggests us to consider some arithmetical relations for the extension B/B_0.

4. We treat the simplest case: $l = 2$, $D = 1$. In the decomposition (2) of B_A^x, each double coset Ux_iB^x corresponds to a left $\mathcal{O}$-ideal $\mathfrak{X}_i = \bigcap \mathcal{O}_\mathfrak{p} x_{i\mathfrak{p}}$, whose right order being $\Gamma_i = \bigcap x_{i\mathfrak{p}}^{-1} \mathcal{O}_\mathfrak{p} x_{i\mathfrak{p}}$ with the unit group

$\Gamma_i = \mathcal{O}_i^x$. H is the class number of B, and the explicit formula is known (for example, in [5]). The result of Theorem 1, for $k=0$ tells us some informations about the behavior of σ on the ideal classes and on the set of types of maximal orders of B. First we try to describe the structure of the finite groups Γ_i/E_0 ($i=1, \cdots, H$) where E_0 is the unit group of F.

Theorem 2. *Γ_i/E_0 is isomorphic to one of the followings; $1, Z_2, Z_3$ (cyclic groups), $\mathfrak{S}_3$ = symmetric group of degree 3, $\mathfrak{A}_4$ = alternate group of degree 4; Let H_j ($j=1,2,3,6,12$) denote the number of i's for which Γ_i/E_0 has order j. They are determined by the following formulae:*

$$\begin{cases} H = H_1 + H_2 + H_3 + H_6 + H_{12} = h(q)\left(\dfrac{\zeta_F(-1)}{2} + \dfrac{h(-q)}{8} + \dfrac{h(-3q)}{6}\right) \\ H_6 = \begin{cases} 0 & \cdots q \equiv 1 \pmod 3 \\ h(q) & \cdots q \not\equiv 1 \pmod 3, \end{cases} \qquad H_{12} = \begin{cases} 0 & \cdots q \equiv 1 \pmod 8 \\ h(q) & \cdots q \not\equiv 1 \pmod 8 \end{cases} \\ 4H_2 + 4H_6 + 2H_{12} = h(q)h(-q), \qquad 4H_3 + 2H_6 + 4H_{12} = h(q)h(-3q) \end{cases}$$

where, $h(n)$ is the class number of $\boldsymbol{Q}(\sqrt{n})$, $\zeta_F(s)$ is the Dedekind zeta function of F. (We exclude the case $q=5$, where $H=1$, $\Gamma_1/E_0 = \mathfrak{A}_5$).

Using these results, we can evaluate $H^1(G, \Gamma_i)$, $G = \mathrm{Gal}(F/\boldsymbol{Q})$ for the self-conjugate unit group $\Gamma_i = {}^\sigma\Gamma_i$:

Proposition 3. (i) *Let U_0 be the unit group of B_{0A}. For the self-conjugate double coset $Ux_iB^x = {}^\sigma(Ux_iB^x)$, we have a bijection*

$$U_0 \backslash Ux_iB^x \cap B_{0A}^x / B_0^x \xrightarrow{\sim} H^1(G, \Gamma_i)$$

ux_ia being mapped to the cocycle defined by $a_\sigma = a\,{}^\sigma a^{-1}$.

(ii) $$\# H^1(G, \Gamma_i) = \begin{cases} 1 & \cdots \text{ if } \Gamma_i/E_0 = \mathfrak{A}_4, \mathfrak{A}_5 \\ 2 & \cdots \text{ otherwise.} \end{cases}$$

From this, we get:

Theorem 3. *Let $\{\mathfrak{X}_1\}, \cdots, \{\mathfrak{X}_H\}$ be the left $\mathcal{O}$-ideal classes where $\mathcal{O}$ is a σ-stable maximal order of B.*

(i) *σ induces a permutation on these classes and the number of fixed points is* $\dim SM_0 = \left[\dfrac{q+19}{24}\right]$.

(ii) *Every self-conjugate class $\{\mathfrak{X}\} = {}^\sigma\{\mathfrak{X}\}$ contains a self-conjugate ideal $\mathfrak{X} = {}^\sigma\mathfrak{X}$.*

(iii) *For the existence of σ-stable maximal order which belongs to the same type as $\mathcal{O}_i$, it is necessary and sufficient that there exists an ideal $\mathfrak{a}$ of F such that $\{\mathfrak{X}_1\mathfrak{a}\} = {}^\sigma\{\mathfrak{X}_i\mathfrak{a}\}$; in particular, the number of types which are σ-stable is also* $\dim SM_0 = \left[\dfrac{q+19}{24}\right]$.

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