27. On Partial Isometries

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1. Introduction. We shall show the relation among paranormality, quasinormality and property of partial isometry and we shall give some necessary conditions on operators which are similar to partial isometries and finally we shall give certain sufficient conditions which imply partial isometries. An operator T means a bounded linear operator on a complex Hilbert space H. The numerical range $W(T)$ is defined by: $W(T)=\{(Tx, x); \|x\|=1, x \in H\}$ and $\overline{W(T)}$ means the closure of $W(T)$. A unitary operator U is said to be *cramped* if its spectrum is contained in some semicircle of the unit circle, that is,

$$\sigma(U)\subset\{e^{i\theta}; \theta_1\leqq\theta\leqq\theta_2, \theta_2-\theta_1<\pi\},$$

where $\sigma(T)$ stands for the spectrum of T. An operator T is said to be *partial isometry* if $T=TT^*T$, *quasinormal* [6] if $TT^*T=T^*TT$, *subnormal* if T has a normal extension and *quasihyponormal* if $\|TTx\| \geqq\|T^*Tx\|$. An operator T is said to be *paranormal* [1], [3], [5], [7] if $\|T^2x\|\cdot\|x\|\geqq\|Tx\|^2$ and *k-paranormal* if $\|x\|^{k-1}\cdot\|T^kx\|\geqq\|Tx\|^k$ [5].

2. Statement of results. Theorem 1. *If T is a partial isometry, then the following six conditions are equivalent:*

(1) *T is quasinormal,*
(2) *T is subnormal,*
(3) *T is hyponormal,*
(4) *T is quasihyponormal,*
(5) *T is paranormal,*
(6) *T is k-paranormal.*

Theorem 1 is an exact precision of Halmos since the equivalent relation between (2) and (3) is cited in the proof of [6, Problem 161].

Corollary 1. *If T is a partial isometry such that T^* is k-paranormal, then T is the direct sum of a co-isometry and zero.*

Corollary 1 is a generalization of the following "*if T is an isometry such that T^* is paranormal, then T is unitary*" [9, Lemma 4].

Theorem 2. *If T is similar to a partial isometry V with the initial projection P and if the similarity could be implemented by an operator A commuting with P, then there exists S satisfying $0\notin\overline{W(S)}$, $SP=PS$ and the following $T^*ST=SP$ hold.*

Corollary 2. *Let T be an operator satisfying the hypothesis of Theorem 2. Then there exist T_1 and S satisfying $0\notin\overline{W(S)}$, $SP=PS$*

and the following $T_1T=P$ *and* $T^*=ST_1S^{-1}$ *hold.*

Corollary 3 [8]. *If* T *is similar to an isometry, then* T *has a left inverse* T_1 *satisfying* $T^*=S^{-1}T_1S$ *for some operator* S *with* $0 \notin \overline{W(S)}$.

Corollary 4. *If* T *is similar to an isometry, then there exists* S *satisfying* $0 \notin \overline{W(S)}$ *and* $T^*ST=S$.

Theorem 2 can be regarded as some converse to the following [2].

Theorem A [2]. *Let* S, T *and* P *be operators such that* S *satisfies* $0 \notin \overline{W(S)}$, P *is a projection commuting with* S *and* $T^*ST=SP$. *Then* T *is similar to a partial isometry with the initial projection* P.

Theorem 3. (i) *Suppose that there exist a normaloid* T_1 *and a projection* P *for a given normaloid* T *such that* $T_1T=P$ *and there exists* S *commuting with* P *satisfying* $0 \notin \overline{W(S)}$ *and* $T^*=S^{-1}T_1S$. *Then* T *is an isometry on* $P(H)$ *the range of* P *which is the subspace of* $N(T)^{\perp}$ *the orthogonal complement of the kernel of* T.

(ii) *In addition to* (i), *if* $TP=T$, *then* T *is a partial isometry with the initial projection* P.

Theorem 3 implies the following theorem.

Theorem B [7]. *If* T *is a left invertible normaloid operator with a left inverse normaloid* T_1 *and if there exists an operator* S *such that* $T^*=S^{-1}T_1S$ *and* $0 \notin \overline{W(S)}$, *then* T *is an isometry.*

Corollary 5. *Let* T *be a normaloid satisfying the all hypotheses in Theorem* 3.

(i) *If* T_1 *is a k-paranormal satisfying* $PT_1=T_1$, *then* T *is the direct sum of a co-isometry and zero.*

(ii) *In addition to* (i), *if* T *is also a k-paranormal, then* T *is the direct sum of a unitary and zero.*

In [4], as some generalizations of [8] and [10] there are given certain sufficient conditions which imply partial isometry operators.

Theorem 4. (i) *Let* $T=UR$ *be an operator with the right-handed polar decomposition and* $T_1=R_1U_1$ *be an operator with the left-handed polar decomposition, where* U *and* U_1 *are partial isometry,* R *and* R_1 *are positive operators respectively. Suppose that there exists a projection* P *satisfying* $P=R_1R$.

(ii) *In addition to* (i), *assume that*

$$T^*=V^*T_1V \quad \text{and} \quad N(R)=N(V^*R_1V)$$

for a cramped unitary operator V *commuting with* P. *Then* R *is the identity operator on* $P(H)$ *which is the subspace of* $N(R)^{\perp}$.

Proofs and detalis will appear in [4] together with some theorems and their applications. Some applications of Theorem 4 are cited in [4]. This paper is early announcement of [4] whose publication delays.

References

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