25. On the Lax-Mizohata Theorem in the Analytic and Gevrey Classes

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1. Introduction. In this paper, we consider the non-characteristic Cauchy problem for the differential operators with analytic or Gevrey coefficients.

L. Boutet de Monvel and P. Krée [2] showed some fundamental properties of analytic and Gevrey symbols of pseudo-differential operators. In [1], L. Hörmander has localized the pseudo-differential operators with analytic symbols in a suitable way on the dual space to extend the regularity and uniqueness theorems, and to study the propagation of the singularities.

Here, using this localized differential operator, we shall give a some necessary relation between the admissible initial data and the number of real roots of the characteristic equation. And, as application of this relation, we extend the Lax-Mizohata theorem to the analytic and Gevrey classes.

A forthcoming paper will give the detailed proof.

2. Definitions and results. Let V be an open set in R^m, we shall denote by $\gamma^{(s)}(V)$ $(s\geqslant 1)$ the set of all $f\in C^\infty(V)$ such that for every compact set $K\subset V$, there are constants C, A with

$$|D^\alpha f(x)|\leqslant CA^{|\alpha|}\alpha!^s, \qquad x\in K, \tag{2.1}$$

for all multi-indexes α. Let $p(x,t;D_x,D_t)=D_t^m+\sum_{j=1}^m a_j(x,t;D_x)D_t^{m-j}$ be a differential operator with coefficients in $\gamma^{(s)}(W)$, where W is a neighborhood of the origin in R^{n+1}, the order of $a_j(x,t;D_x)$ is less than j, and

$$D_t=\frac{1}{i}\frac{\partial}{\partial t},\quad D_x=\left(\frac{1}{i}\frac{\partial}{\partial x_1},\cdots,\frac{1}{i}\frac{\partial}{\partial x_n}\right),\quad x=(x_1,\cdots,x_n).$$

We shall denote by $p_0(x,t;\xi,\lambda)$ the principal symbol of $p(x,t;D_x,D_t)$.

Theorem 2.1. *Suppose that $p_0(0,0;\hat{\xi},\lambda)=0$ $(|\hat{\xi}|\neq 0)$ has μ real and ν non-real roots $(\mu+\nu=m)$, and u is a C^∞-solution of the equation $p(x,t;D_x,D_t)u=0$ defined in a neighborhood of the origin such that $D_t^ju(x,0)=0$ for $0\leqslant j\leqslant\mu-1$. Then $(0,\hat{\xi})$ is in the complement of wave front set $WF_s(D_t^\mu u(x,0))$,* i.e. *there are a neighborhood U of 0, a conic neighborhood Γ of $\hat{\xi}$, and a bounded sequence $u_N\in\mathcal{E}'(R^n)$ which is equal to $D_t^\mu u(x,0)$ in U such that*

$$|\hat{u}_N(\xi)|\leqslant C(CN^s)^N|\xi|^{-N} \tag{2.2}$$

is valid for some constant C when $\xi\in\Gamma$.

Consider the following problem

$$(\mathrm{P})_{\mathrm{k}} \qquad \begin{cases} p(x,t;D_x,D_t)u=0 \\ D_t^j u(x,0)=u_j(x)\ 0\leqslant j\leqslant k-1\ (k\leqslant m), \end{cases}$$

then by the Theorem 2.1, we have

Corollary 2.1. *If the problem* $(\mathrm{P})_{\mathrm{k}}$ *has a* C^∞*-solution in a neighborhood of the origin for any given* $(u_0(x),\cdots,u_{k-1}(x))\in\prod^k C^\infty(R^n)$*, then* $p_0(0,0;\xi,\lambda)=0$ *must have more than k real roots for every* $\xi\neq 0$.

We shall say that the Cauchy problem $(\mathrm{P})_{\mathrm{m}}$ is $\gamma^{(s)}$-well posed in a neighborhood of the origin ($s>1$), if there exists a neighborhood D of 0 in R^{n+1} such that the problem

$$(2.3) \qquad \begin{cases} p(x,t;D_x,D_t)u=0 \text{ in } D \\ D_t^j u(x,0)=u_j(x)\ 0\leqslant j\leqslant m-1, \text{ in } D\cap(t=0) \end{cases}$$

has a unique solution $u\in C^\infty(D)$ for any given initial data $(u_0(x),\cdots,u_{m-1}(x))\in\prod^m\gamma^{(s)}(R^n)$. Then Theorem 2.1 and the Baire's category theorem show

Theorem 2.2. *Let s be* >1*. Then, for the Cauchy problem* $(\mathrm{P})_{\mathrm{m}}$ *to be* $\gamma^{(s)}$*-well posed in a neighborhood of the origin, it is necessary that* $p_0(0,0;\xi,\lambda)=0$ *has only real roots for any* $\xi\neq 0$.

Theorem 2.3 (c.f. [4])**.** *Suppose that* $s=1$*, and* $p_0(0,0;\hat{\xi},\lambda)=0$ $(|\hat{\xi}|\neq 0)$ *has at least one non-real root. Then there exists an open neighborhood U of the origin in* R^n *such that for any open neighborhood W of* 0 *in* R^{n+1} *satisfying* $W\cap(t=0)=U$*, there is an analytic initial data on U for which the solution of the Cauchy problem* $(\mathrm{P})_{\mathrm{m}}$ *cannot be continued analytically whole in W.*

3. Proof of Theorem 2.1. Let W be an open set in R^{n+1}, and Γ be a conic set in $R^{n+1}\backslash 0$. We write $y=(x,t)$, $\eta=(\xi,\lambda)$ and $|\eta|^2=|\xi|^2+|\lambda|^2$. Following [2], we shall say that the formal sum $p=\sum_{k=0}^\infty p_k(y,\eta)$ is a symbol on $W\times\Gamma$ of class s with order (r,t), if each $p_k(y,\eta)$ is a smooth function on $W\times\Gamma$, homogeneous degree $r+t-k$ with respect to η and there exists constants C,A such that for any integer k, any multi-indexes α,β, and any $(y,\eta)\in W\times\Gamma$, the following inequality holds

$$(3.1) \qquad |p_{k(\alpha)}^{(\beta)}(y,\eta)|\leqslant CA^{k+|\alpha+\beta|}|\eta|^t|\xi|^{r-k-|\beta|}(k+|\alpha|)!^s\beta!,$$

where

$$p_{k(\alpha)}^{(\beta)}(y,\eta)=\left(\frac{1}{i}\frac{\partial}{\partial y}\right)^\alpha\left(\frac{\partial}{\partial\eta}\right)^\beta p_k(y,\eta).$$

Lemma 3.1. *Suppose that* $p_0(0,0;\hat{\xi},\lambda)=0$ $(|\hat{\xi}|\neq 0)$ *has* μ *real and* ν *non-real roots* $(\mu+\nu=m)$*. Then there are a neighborhood W of* 0 *in* R^{n+1}*, a conic neighborhood* Γ *of* $\hat{\xi}$ *in* $R^n\backslash 0$ *and symbols* a^j $(1\leqslant j\leqslant\mu)$*,* b^i $(1\leqslant i\leqslant\nu)$ *on* $W\times(\Gamma\times R)$ *which are independent of* λ*, of class s with order* $(j,0)$*,* $(i,0)$ *respectively, and satisfy the equation*

$$(3.2) \qquad p(x,t;\xi,\lambda)=\left(\lambda^\mu+\sum_{j=1}^{\mu}a^j(x,t;\xi)\lambda^{\mu-j}\right)\circ\left(\lambda^\nu+\sum_{i=1}^{\nu}b^i(x,t;\xi)\lambda^{\nu-i}\right)$$

as symbols on $W\times(\Gamma\times R)$ *(for the composition of symbols, see* [2]*). Here,* $\lambda^{\mu}+\sum_{j=1}^{\mu}a_0^j(0,0;\hat{\xi})\lambda^{\mu-j}=0$, $\lambda^{\nu}+\sum_{i=1}^{\nu}b_0^i(0,0;\hat{\xi})\lambda^{\nu-i}=0$ *has only real and non-real roots respectively.*

Corollary 3.1. *Under the same condition in Lemma* 3.1, *there exists a neighborhood* W *of* 0 *in* R^{n+1}, *a conic neighborhood* Γ *in* $R^n\backslash 0$ *and symbols* q, r *on* $W\times(\Gamma\times R)$ *which satisfy the followings,* i.e. $p\circ q=r$, *where* $r=\lambda^{\mu}+\sum_{j=1}^{\mu}a^j(y,\xi)\lambda^{\mu-j}$ *is the same one in Lemma* 3.1 *and* q *is of class* s *with order* $(0,-\nu)$. *Moreover, for* $k+|\beta|\geqslant 1$, $(y,\eta)\in W\times(\Gamma\times R)$, *the inequality*

$$|q_{k(\alpha)}^{(\beta)}(y,\eta)|\leqslant CA^{k+|\alpha+\beta|}|\eta|^{-\nu-1}|\xi|^{1-k-|\beta|}(k+|\alpha|)!^s\beta! \tag{3.3}$$

holds.

Using Corollary 3.1, we can prove Theorem 2.1.

References

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