24. Hölder Conditions for the Local Times of Certain Gaussian Processes with Stationary Increments

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1. Let $\{X(t,\omega);\ -\infty<t<\infty\}$ be a path continuous centered stationary Gaussian process with the spectral density function $f(\lambda)$ given by

$$f(\lambda)=a^{2\alpha}\frac{\Gamma(\alpha+1/2)}{\Gamma(1/2)\Gamma(\alpha)}(\lambda^2+a^2)^{-(\alpha+1/2)},\qquad 0<\alpha<1/2.$$

Then owing to Berman's result [2], there exists the local time $\psi(x,t,\omega)$ of $X(t,\omega)$ which is jointly continuous in x and t almost surely. For the local Hölder conditions of this local time, Davies [3] has proved the following:

$$0<c_1\le\overline{\lim_{h\downarrow 0}}\frac{|\psi(X(t),t+h,\omega)-\psi(X(t),t,\omega)|}{h^{1-\alpha}(\log\log 1/h)^{\alpha}}\le c_2<+\infty$$

for almost all ω.

We will extend his result to more wide class of Gaussian processes with stationary increments. We will give not only a local Hölder condition but also a uniform Hölder condition with respect to the upper bound. As for the lower bound, it is still open problem for our class.

2. Let $\{X(t,\omega);\ 0\le t\le 1\}$ be a path continuous centered Gaussian process with stationary increments: $E(X(t)-X(s))^2=\sigma^2(|t-s|)$. We assume the following:

(1) $\sigma(x)$ is a non-decreasing continuous nearly regular varying function with index α, $0<\alpha<1$, i.e. there exist two positive constants c and c', and also a slowly varying function $s(x)$ such that

$$cx^{\alpha}s(x)\le\sigma(x)\le c'x^{\alpha}s(x),$$

(2) $x/\sigma(x)$ is non-decreasing,

(3) $\sigma(x)$ is differentiable for $x>0$ with the derivative $\sigma'(x)$ such that

$$\sigma'(x)\le\beta\sigma(x)/x,\quad \beta<1,\quad x>0.$$

(4) Denote by A_{2n} the correlation matrix $(r_{i,j})_{i,j=1}^{2n}$:

$$r_{i,j}=E\left[\frac{(X(t_i)-X(t_{i-1}))(X(t_j)-X(t_{j-1}))}{\sigma(t_i-t_{i-1})\sigma(t_j-t_{j-1})}\right],\qquad i,j=1,\cdots,2n,$$

for a partition $0=t_0<t_1<\cdots<t_{2n}\le 1$. Then there exist a positive constant c_3 and a positive integer n_0 such that

$$\det A_{2n}\ge c_3^{2n}$$

holds for any partition of $[0,1]$ and any $n\ge n_0$.

We notice that if $\sigma^2(x)$ is a differentiable and concave nearly regular varying function with index $0<2\alpha\leq 1$, then all conditions (1) to (4) are fulfilled. By Berman's result (Lemma 6.1 of [1]), our conditions (1) and (4) garantee the existence of jointly continuous local time $\psi(x, t, \omega)$ for almost all ω.

Theorem. *Under the conditions* (1) *to* (4),

(i) $$\overline{\lim_{h\downarrow 0}} \frac{|\psi(X(t), t+h, \omega)-\psi(X(t), t, \omega)|}{h/\sigma(h/\log\log 1/h)} \leq c_4 < +\infty \text{ a.s.},$$

(ii) $$\overline{\lim_{h\downarrow 0}} \frac{|\psi(x, t+h, \omega)-\psi(x, t, \omega)|}{h/\sigma(h/\log\log 1/h)} \leq c_5 < +\infty \text{ a.s.},$$

(iii) $$\overline{\lim_{\substack{|t-s|\downarrow 0\\ 0\leq t,s\leq 1}}} \frac{|\psi(x, t, w)-\psi(x, s, \omega)|}{|t-s|/\sigma(|t-s|/\log 1/|t-s|)} \leq c_6 < +\infty \text{ a.s.}$$

3. First we prove the following lemma.

Lemma. *Let $\sigma(x)$ be a function satisfying the conditions* (1) *to* (3). *Then*

$$I_n=\int_{0<t_1<\cdots<t_n<h} 1\Big/\prod_{j=1}^n \sigma(t_j-t_{j-1})dt_1\cdots dt_n \ (t_0=0)$$
$$\leq \frac{(2h)^n}{(1-\beta)^n n!}\Big(1\Big/\prod_{j=1}^n \sigma(h/j)\Big).$$

Proof. Changing the variables $t_1, \cdots, t_n$ of integration to $u_1, \cdots, u_n$ such that

$$u_j=(t_j-t_{j-1})/(h-t_{j-1}), \qquad 2\leq j\leq n$$

and

$$u_1=t_1/h,$$

we have

$$I_n=h^n\int_0^1\cdots\int_0^1 \prod_{j=1}^n ((1-u_j)^{n-j}/\sigma(u_j(1-u_{j-1})\cdots(1-u_1)h))du_1\cdots du_n.$$

By the assumption (3) and integration by part,

$$\int_0^1 \frac{du_n}{\sigma(u_n(1-u_{n-1})\cdots(1-u_1)h)} \leq \frac{1}{(1-\beta)\sigma((1-u_{n-1})\cdots(1-u_1)h)}.$$

By induction if we get the inequality:

$$\int_0^1\cdots\int_0^1 \prod_{j=n-k+1}^n \frac{(1-u_j)^{n-j}}{\sigma(u_j(1-u_{j-1})\cdots(1-u_1)h)}du_n\cdots du_{n-k+1}$$
$$\leq \frac{2^k}{(1-\beta)^k k!}\Big(1\Big/\prod_{j=1}^k \sigma((1-u_{n-k})\cdots(1-u_1)h/j)\Big),$$

then we have

$$\int_0^1\cdots\int_0^1 \prod_{j=n-k}^n \frac{(1-u_j)^{n-j}}{\sigma(u_j(1-u_{j-1})\cdots(1-u_1)h)}du_n\cdots du_{n-k}$$
$$\leq \frac{2^k}{(1-\beta)^k k!}\Big(\int_0^{1/(k+1)}+\int_{1/(k+1)}^1\Big)$$
$$\times \frac{(1-u_{n-k})^k}{\sigma(u_{n-k}(1-u_{n-k-1})\cdots(1-u_1)h)}\Big/\prod_{j=1}^k \sigma((1-u_{n-k})\cdots(1-u_1)h/j)du_{n-k}$$
$$=J_1+J_2.$$

By the assumptions (1), (2), (3) and integrations by part,

$$J_1 \le \frac{2^k}{(1-\beta)^k k!}\Big(1\Big/\prod_{j=1}^{k}\sigma((1-u_{n-k-1})\cdots(1-u_1)h/j)\Big)$$
$$\times\int_0^{1/(k+1)}\frac{du_{n-k}}{\sigma(u_{n-k}(1-u_{n-k-1})\cdots(1-u_1)h)}$$
$$\le\frac{2^k}{(1-\beta)^{k+1}(k+1)!}\Big(1\Big/\prod_{j=1}^{k+1}\sigma((1-u_{n-k-1})\cdots(1-u_1)h/j)\Big),$$

and

$$J_2 \le \frac{2^k}{(1-\beta)^k k!\,\sigma((1-u_{n-k-1})\cdots(1-u_1)h/(k+1))}$$
$$\times\int_0^{k/(k+1)} v_{n-k}^k\Big/\prod_{j=1}^{k}\sigma(v_{n-k}(1-u_{n-k-1})\cdots(1-u_1)h/j)dv_{n-k}$$
$$\le\frac{2^k}{(1-\beta)^k k!\,(k+1-k\beta)}\Big/\prod_{j=1}^{k+1}\sigma((1-u_{n-k-1})\cdots(1-u_1)h/j).$$

It follows therefore that

$$J_1+J_2\le\frac{2^{k+1}}{(1-\beta)^{k+1}(k+1)!}\Big(1\Big/\prod_{j=1}^{k+1}\sigma((1-u_{n-k-1})\cdots(1-u_1)h/j)\Big).$$

This gives the proof of the lemma.

Now we prove (i) of Theorem. According as Davies [(27) of 3] and by our lemma, we have

$$E\,|\psi(X(t),t+h,\omega)-\psi(X(t),t,\omega)|^{2n}$$
$$=(2\pi)^{-n}\int_t^{t+h}\cdots\int_t^{t+h}(\det A_{2n})^{-1/2}\Big/\prod_{j=1}^{2n}\sigma(s_j-s_{j-1})ds_1\cdots ds_{2n}$$
$$\le\frac{(2n)!}{(2\pi c_3)^n}\int_{0<t_1<\cdots<t_{2n}<h}1\Big/\prod_{j=1}^{2n}\sigma(t_j-t_{j-1})dt_1\cdots dt_{2n}$$
$$\le\frac{(\sqrt{2}\,h)^{2n}}{(\sqrt{\pi c_3}(1-\beta))^{2n}}\Big(1\Big/\prod_{j=1}^{2n}\sigma(h/j)\Big).$$

Since $1/\sigma(1/x)$ is a nearly regular varying function with index α at infinity, applying Theorem 1 of [4] to the positive random variable

$$X=|\psi(X(t),t+h,\omega)-\psi(X(t),t,\omega)|/h,$$

there exists $n_0>0$ such that

$$P(X\ge A/\sigma(1/x))\le e^{-\alpha hx/2}$$

holds for all $hx\ge n_0$. Setting

$$x=\frac{4\log\log 1/h_n}{\alpha h_n},\quad\text{and}\quad h_n=e^{-n},$$

$$P\Big(|\psi(X(t),t+h_n,\omega)-\psi(X(t),t,\omega)|>Ah_n/\sigma\Big(\frac{\alpha h_n}{4\log\log 1/h_n}\Big)\Big)\le\frac{1}{n^2}$$

holds for large n. Finally, by standard argument using Borel-Cantelli lemma and non-decreasingness of $\psi(x,t,\omega)$ in t, we have

$$\overline{\lim_{h\downarrow 0}}\frac{|\psi(X(t),t+h,\omega)-\psi(X(t),t,\omega)|}{h/\sigma(h/\log\log 1/h)}\le c_4<+\infty\ \text{a.s.}$$

The proof of (ii) is just the same way as that of (i), so we omit it.

To prove (iii), using the same argument as above it follows that there exists $m_0(\omega)$ with probability 1 such that for all $m \geq m_0(\omega)$ and $k=1, \cdots, 2^m$

$$|\psi(x, k2^m, \omega)-\psi(x,(k-1)2^{-m}, \omega)| \leq A2^{-m}/\sigma\left(\frac{2^{-m-2}\alpha}{m \log 2}\right).$$

Since $\psi(x, t, \omega)$ is continuous in t, we have

$$\begin{aligned}|\psi(x, t, \omega)-\psi(x, s, \omega)| &\leq \sum_{k=m}^{\infty}\left|\psi\left(x, \sum_{j=m+1}^{k+1} \varepsilon_j 2^{-j}+\bar{k}2^{-m}, \omega\right)-\psi\left(x, \sum_{j=m+1}^{k} \varepsilon_j 2^{-j}+\bar{k}2^{-m}, \omega\right)\right| \\ &\quad+\sum_{k=m}^{\infty}\left|\psi\left(x, \sum_{j=m+1}^{k+1} \varepsilon'_j 2^{-j}+\bar{k}'2^{-m}, \omega\right)-\psi\left(x, \sum_{j=m+1}^{k} \varepsilon'_j 2^{-j}+\bar{k}'2^{-m}, \omega\right)\right| \\ &\quad+|\psi(x, \bar{k}2^{-m}, \omega)-\psi(x, \bar{k}'2^{-m}, \omega)|\end{aligned}$$

for

$$2^{-m-1} \leq t-s<2^{-m}, \qquad s=\bar{k}2^{-m}+\sum_{j=m+1}^{\infty} \varepsilon_j 2^{-j}$$

$$t=\bar{k}'2^{-m}+\sum_{j=m+1}^{\infty} \varepsilon'_j 2^{-j}, \ \varepsilon_j \text{ and } \varepsilon'_j=0 \text{ or } 1,\ 0 \leq \bar{k}'-\bar{k} \leq 1.$$

By the assumption (1), it follows that

$$\begin{aligned}\sum_{k=m}^{\infty} 2^{-k}/\sigma\left(\frac{2^{-k-2}\alpha}{k \log 2}\right) &\leq C2^{-m}/\sigma\left(\frac{2^{-m-2}\alpha}{m \log 2}\right) \\ &\leq C'\,|t-s|/\sigma(|t-s|/\log 1/|t-s|).\end{aligned}$$

Therefore we have

$$\varlimsup_{\substack{|t-s| \downarrow 0 \\ 0 \leq t, s \leq 1}} \frac{|\psi(x, t, \omega)-\psi(x, s, \omega)|}{|t-s|/\sigma(|t-s|/\log 1/|t-s|)} \leq c_6<+\infty \text{ a.s.}$$

References

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